Solutions 3

5.3 Lemma

\[ C \vdash \forall \overline{r} \psi [\overline{a}] \iff \text{for all } b \in \text{dom}(\overline{a}), C \vdash \psi[^*](\overline{r}/b). \]

Proof:

\[ C \vdash \forall \overline{r} \psi [\overline{a}] \iff C \vdash \neg \exists \overline{r} \neg \psi [\overline{a}] \iff \neg C \vdash \exists \overline{r} \neg \psi [\overline{a}] \]

\[ \iff \text{there exists } b \in A \text{ such that } C \vdash \neg \psi[^*](\overline{r}/b) \]

\[ \iff \text{for all } b \in A, \neg C \vdash \neg \psi[^*](\overline{r}/b) \]

\[ \iff \text{for all } b \in A, \neg (\neg C \vdash \psi[^*](\overline{r}/b)) \]

\[ \iff \text{for all } b \in A, C \vdash \psi[^*](\overline{r}/b). \]

Exercise 5.4 (i).

What happens if we substitute \( v_2 \) (or \( v_3 \)) for \( v_i \) in the formula Prime \((v_i)\)?

Solution:

Prime \((v_i)\) is the formula \((\phi \land \neg F_i(v_i, v_i) \equiv v_i)\) where \( \phi \) is the formula \( \forall v_2 \forall v_3 \left(F_i(v_i, v_i) \equiv v_i \rightarrow (v_2 \equiv v_i \lor v_3 \equiv v_i)\right)\).

Substituting \( v_2 \) for \( v_i \) here we obtain

\[ \phi \vdash \forall v_2 \forall v_3 \left(F_i(v_i, v_i) \equiv v_i \rightarrow (v_2 \equiv v_i \lor v_3 \equiv v_i)\right) \]

which is a tautology -- i.e., \( C \vdash \phi[^*] \) for all \( C \) and all \( \overline{a} \in A^n \).

Hence, Prime \((v_i)\) is (equivalent in all structures to)

just \( \neg F_i(v_i, v_i) \equiv v_i \). So for \( C = \langle \langle v_{i_0}, v_i \rangle, \rangle \)

we have \( C \vdash \phi[^*] \) iff \( a_2 \cdot a_2 \neq a_2 \), i.e., \( a_2 \neq 1 \).

Similarly, for \( v_3 \) in place of \( v_i \).
Exercise 5.5

Let \( \mathcal{A} \) be the structure \( \langle \mathbb{Z}^n ; +, \cdot \rangle \). What is the type \( \sigma \) of \( \mathcal{A} \)? Write down a formula in this language - \( \psi \) say - having the property that for all \( \vec{a} \in A^n \), \( \mathcal{A} \models \psi[\vec{a}] \iff a_1 \geq 0 \).

**Solution**

\[ \sigma = \langle \emptyset, \{1, 2\}, \emptyset, \emptyset, \emptyset \rangle, \text{ where } \mu(1) = \mu(2) = 2. \]

We use Lagrange's Theorem: every non-negative integer is the sum of four squares of integers.

Thus, we take \( \psi \) to be

\[ \forall v_2 \forall v_3 \forall v_4 \forall v_5 \quad v_1 = F_1\left(F_1\left(F_2\left(v_2, v_3\right), F_2\left(v_4, v_5\right)\right), F_1\left(F_2\left(v_4, v_5\right), F_2\left(v_2, v_3\right)\right)\right). \]

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Exercise 5.6

We prove, by induction on \( \psi \), that for all \( \vec{a}, \vec{b} \in A^n \), if \( a_1^i = b_1^i \) for all those \( v_i \) such that \( v_i \) occurs free in \( \psi \), then \( \mathcal{A} \models \psi[\vec{a}] \iff \mathcal{A} \models \psi[\vec{b}] \).

**Proof.**

If \( \psi \) has the form \( \tau_1 = \tau_2 \), then by 4.9.2(i) and 4.5, \( \tau_1[\vec{a}] = \tau_2[\vec{b}] \) for \( r = 1, 2 \). It follows that \( \tau_1[\vec{a}] = \tau_2[\vec{a}] \iff \tau_1[\vec{b}] = \tau_2[\vec{b}] \), i.e. \( \mathcal{A} \models \psi[\vec{a}] \iff \mathcal{A} \models \psi[\vec{b}] \).

If \( \psi \) has the form \( \psi_i (\tau_1, \ldots, \tau_p(i)) \) it follows by 4.9.2(ii) and 4.5 that \( \tau_v[\vec{a}] = \tau_v[\vec{b}] \) for \( v = 1, \ldots, p(i) \). Hence \( \langle \tau_1[\vec{a}], \ldots, \tau_p(i) \rangle \in R_i \iff \langle \tau_1[\vec{b}], \ldots, \tau_p(i)[\vec{b}] \rangle \in R_i \), i.e. \( \mathcal{A} \models \psi[\vec{a}] \iff \mathcal{A} \models \psi[\vec{b}] \).

If \( \psi \) has the form \( (\psi_1 \land \psi_2) \) then by 4.10.1(ii)(b) we
have that $a_q = b_q$ for all $q$, s.t. $v_q$ occurs free in $\psi$, so by the inductive hypothesis $G \models \psi[a]$ iff $G \models \psi[b]$, and similarly $G \models X[a]$ iff $G \models X[b]$. It follows that $G \models (\psi_1 X)[a]$ iff $G \models (\psi_1 X)[b]$.

The proof is similar if $\phi$ has the form $\neg \psi$ by using 4.10.1(ii)(c).

The interesting case is when $\phi$ has the form $\exists y \psi$. Our hypotheses — that $a_q = b_q$ for all $q$, s.t. $v_q$ occurs free in $\psi$ — implies (by 4.10.1(ii)(c)) only that $a_q = b_q$ for all $q$, s.t. $v_q$ occurs free in $\psi$ except, possibly, for $q = p$.

Now $G \models \phi[a]$ iff for some $d \in A$, $G \models \psi[\overline{a}(p/d)]$. But the sequences $\overline{a}(p/d)$ and $\overline{b}(p/d)$ agree at all $q$, s.t. $v_q$ occurs free in $\psi$ (they now agree in the $p$'th coordinate), so we may indeed apply the inductive hypothesis to infer that (for some $d \in A$) $G \models \psi[\overline{b}(p/d)]$. It follows that $G \models \exists y \psi[b]$, i.e. $G \models \phi[b]$ as required.

The implication $G \models \phi[i] \Rightarrow G \models \phi[\overline{a}]$ is proved in an identical way.