Solutions 3

5.3 *Lemma*

$\mathfrak{A} \models \forall v_p \psi [\bar{a}] \iff$ for all $b \in dom(\mathfrak{A})$, $\mathfrak{A} \models \psi[\bar{a}(p/b)]$.

Proof.

$\mathfrak{A} \models \forall v_p \psi[\bar{a}] \iff \mathfrak{A} \models \neg \exists v_p \neg \psi [\bar{a}] \iff$ not $\mathfrak{A} \models \exists v_p \neg \psi [\bar{a}]$

$\iff$ not (there exists $b \in A$ such that $\mathfrak{A} \models \neg \psi[\bar{a}(p/b)]$)

$\iff$ for all $b \in A$, not $\mathfrak{A} \models \neg \psi[\bar{a}(p/b)]$

$\iff$ for all $b \in A$, not(not $\mathfrak{A} \models \psi[\bar{a}(p/b)]$

$\iff$ for all $b \in A$, $\mathfrak{A} \models \psi[\bar{a}(p/b)]$.

$\square$

Exercise 5.4 (1).

What happens if we substitute v_2 (or v_3) for v_1 in the formula Prime (v_1)?

Solution

Prime (v_1) is the formula $(\phi \wedge \neg F_1(v_1, v_1) \doteq v_1)$ where ϕ is the formula $\forall v_2 \forall v_3 (F_1(v_2, v_3) \doteq v_1 \to (v_2 \doteq v_1 \vee v_3 \doteq v_1))$.

Substituting v_2 for v_1 here we obtain

ϕ_2: $\forall v_2 \forall v_3 (F_1(v_2, v_3) \doteq v_2 \to (v_2 \doteq v_2 \vee v_3 \doteq v_2))$

which is a tautology – i.e. $\mathfrak{A} \models \phi_2[\bar{a}]$ for all $\mathfrak{A}$ and all $\bar{a} \in A^\omega$.

Hence, Prime(v_2) is (equivalent in all structures to) just $\neg F_1(v_2, v_2) \doteq v_2$. So for $\mathfrak{A} = \langle \mathbb{N}_{>0}; \cdot \rangle$ we have $\mathfrak{A} \models \phi_2[\bar{a}]$ iff $a_2 \cdot a_2 \neq a_2$, i.e. $a_2 \neq 1$.

Similarly for v_3 in place of v_1.

Exercise 5.5

Let $\mathfrak{A}$ be the structure $\langle \mathbb{Z}; +, \cdot \rangle$. What is the type σ of $\mathfrak{A}$? Write down a formula in this language – ψ say – having the property that for all $\bar{a} \in A^\omega$, $\mathfrak{A} \models \psi[\bar{a}] \iff a_1 \geq 0$.

Solution

$\sigma = \langle \emptyset; \{1,2\}; \emptyset, \emptyset, \mu \rangle$, where $\mu(1) = \mu(2) = 2$.

We use Lagrange's Theorem: every non-negative integer is the sum of four squares of integers.
Thus, we take ψ to be

$$\exists v_2 \exists v_3 \exists v_4 \exists v_5 \;\; v_1 \doteq F_1(F_1(F_2(v_2,v_2), F_2(v_3,v_3)), F_1(F_2(v_4,v_4), F_2(v_5,v_5))).$$

Exercise 5.6

We prove, by induction on ϕ, that for all $\bar{a}, \bar{b} \in A^\omega$, if $a_q = b_q$ for all those q such that v_q occurs free in ϕ, then $\mathfrak{A} \models \phi[\bar{a}] \iff \mathfrak{A} \models \phi[\bar{b}]$.

Proof.

- If ϕ has the form $\tau_1 \doteq \tau_2$, then by 4.9.2 (ii) and 4.5, $\tau_r^{\mathfrak{A}}[\bar{a}] = \tau_r^{\mathfrak{A}}[\bar{b}]$ for $r = 1, 2$. It follows that $\tau_1^{\mathfrak{A}}[\bar{a}] = \tau_2^{\mathfrak{A}}[\bar{a}]$ iff $\tau_1^{\mathfrak{A}}[\bar{b}] = \tau_2^{\mathfrak{A}}[\bar{b}]$, i.e. $\mathfrak{A} \models \phi[\bar{a}]$ iff $\mathfrak{A} \models \phi[\bar{b}]$.

- If ϕ has the form $P_i(\tau_1, \ldots, \tau_{\rho(i)})$ it follows by 4.9.2. (ii) and 4.5 that $\tau_r^{\mathfrak{A}}[\bar{a}] = \tau_r^{\mathfrak{A}}[\bar{b}]$ for $r = 1, \ldots, \rho(i)$. Hence $\langle \tau_1^{\mathfrak{A}}[\bar{a}], \ldots, \tau_{\rho(i)}^{\mathfrak{A}} \rangle \in R_i$ iff $\langle \tau_1^{\mathfrak{A}}[\bar{b}], \ldots, \tau_{\rho(i)}^{\mathfrak{A}}[\bar{b}]] \rangle \in R_i$, i.e. $\mathfrak{A} \models \phi[\bar{a}]$ iff $\mathfrak{A} \models \phi[\bar{b}]$.

- If ϕ has the form $(\psi \wedge \chi)$ then by 4.10.1 (ii)(b) we have that $a_q = b_q$ for all q s.th. v_q occurs free in ψ, so by the inductive hypothesis $\mathfrak{A} \models \psi[\bar{a}]$ iff $\mathfrak{A} \models \psi[\bar{b}]$, and similarly $\mathfrak{A} \models \chi[\bar{a}]$ iff $\mathfrak{A} \models \chi[\bar{b}]$. It follows that $\mathfrak{A} \models (\psi \wedge \chi)[\bar{a}]$ iff $\mathfrak{A} \models (\psi \wedge \chi)[\bar{b}]$.

- The proof is similar if ϕ has the form $\neg\psi$ by using 4.10.1 (ii) (c).

- The interesting case is when ϕ has the form $\exists v_p \psi$. Our hypothesis – that $a_q = b_q$ for all q such that v_q occurs free in ϕ – implies (by 4.10.1 (ii) (d1)) only that $a_q = b_q$ for all q s.th. v_q occurs free in ψ except, possibly, for $q = p$.
  Now $\mathfrak{A} \models \phi[\bar{a}]$ iff for some $d \in A$, $\mathfrak{A} \models \psi[\bar{a}(p/d)]$.
  But the sequences $\bar{a}(p/d)$ and $\bar{b}(p/d)$ agree at <u>all</u> q s.th. v_q occurs free in ψ (they now agree in the p'th coordinate), so we may indeed apply the inductive hypothesis to infer that (for some $d \in A$) $\mathfrak{A} \models \psi[\bar{b}(p/d)]$. It follows that $\mathfrak{A} \models \exists v_p \psi[\bar{b}]$, i.e. $\mathfrak{A} \models \phi[\bar{b}]$ as required.
  The implication $\mathfrak{A} \models \phi[\bar{b}] \Rightarrow \mathfrak{A} \models \phi[\bar{a}]$ is proved in an identical way.
