Solutions 2

3.10 Exercise

Prove that any embedding can be decomposed into an isomorphism and an identity embedding. I.e. show that if $\mathfrak{A}, \mathfrak{B} \in K_\sigma$ and $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$, then there exists $\mathfrak{A}^* \in K_\sigma$ such that $\mathfrak{A}^* \subseteq \mathfrak{B}$ and $\pi : \mathfrak{A} \cong \mathfrak{A}^*$.

Solution

Let (with the usual notation for $\mathfrak{A}, \mathfrak{B}$) $A^* := \{\pi(a) : a \in A\}$. Then A^* is closed under each g_j $(j \in J)$. For if $\pi(a_1), \ldots, \pi(a_{\mu(j)}) \in A^*$ (for $a_1, \ldots, a_{\mu(j)} \in A$) then $g_j(\pi(a_1), \ldots, \pi(a_{\mu(j)})) = \pi(f_j(a_1, \ldots, a_{\mu(j)}))$ (by 2.4(c) since

$\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$). Since $f_j(a_1, \ldots, a_{\mu(j)}) \in A$ this shows that $g_j(\pi(a_1), \ldots, \pi(a_{\mu(j)})) \in A^*$.

A similar argument shows that $d_k \in A^*$ for each $k \in K$. Thus, A^* is the domain of a unique substructure, call it $\mathfrak{A}^*$, of $\mathfrak{B}$ by 3.3. The relations, functions and distinguished elements of $\mathfrak{A}^*$ are necessarily given by 3.1 (with A^* in place of A) from which it immediately follows that $\pi : \mathfrak{A} \hookrightarrow \mathfrak{A}^*$ (since $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$). But π obviously maps onto A^*. So $\pi : \mathfrak{A} \cong \mathfrak{A}^*$ as required.

4.5 Lemma.

With the usual notation, if $\bar{a}, \bar{b} \in A^\omega$ (say $\bar{a} = \langle a_1, a_2, \ldots \rangle$, $\bar{b} = \langle b_1, b_2, \ldots \rangle$) and $a_p = b_p$ for all those p such that v_p occurs in τ, then $\tau^{\mathfrak{A}}[\bar{a}] = \tau^{\mathfrak{A}}[\bar{b}]$.

Proof.

We use induction on τ. If τ is c_k then $\tau^{\mathfrak{a}}[\bar{a}] = e_k = \tau^{\mathfrak{a}}[\bar{e}]$. If τ is v_p then $\tau^{\mathfrak{a}}[\bar{a}] = a_p$ and $\tau^{\mathfrak{a}}[\bar{e}] = e_p$. But v_p occurs in v_p so by the lemma hypothesis we have $a_p = e_p$, i.e. $\tau^{\mathfrak{a}}[\bar{a}] = \tau^{\mathfrak{a}}[\bar{e}]$.

Now suppose that τ is $F_j(\tau_1, \ldots, \tau_{\mu(j)})$ (for some $j \in J$) and that the lemma is true for $\tau_1, \ldots, \tau_{\mu(j)}$. Suppose that $a_p = e_p$ for each p such that v_p occurs in τ.

Let $1 \leq r \leq \mu(j)$. Now for any p, if v_p occurs in τ_r then it occurs in τ (by definition of "occuring" — see bottom of page 14), so $a_p = e_p$. Hence, by the inductive hypothesis, $\tau_r^{\mathfrak{a}}[\bar{a}] = \tau_r^{\mathfrak{a}}[\bar{e}]$.

This is true for each $r = 1, \ldots, \mu(j)$.

Therefore, $f_j(\tau_1^{\mathfrak{a}}[\bar{a}], \ldots, \tau_{\mu(j)}^{\mathfrak{a}}[\bar{a}]) = f_j(\tau_1^{\mathfrak{a}}[\bar{e}], \ldots, \tau_{\mu(j)}^{\mathfrak{a}}[\bar{e}])$.

I.e. $\tau^{\mathfrak{a}}[\bar{a}] = \tau^{\mathfrak{a}}[\bar{e}]$ (by 4.4 (iii)), as required.

4.8 Exercise

Describe in a mathematical way the collection of functions $\{\tau^{\mathfrak{a}} : A \to A : \tau \in \mathcal{T}_1\}$, where $\mathcal{T}_1$ denotes the set of terms in which at most the variable v_1 occurs, in the following cases:

(1) $\mathfrak{a} = \langle \mathbb{R}; +, \cdot, -; 0, 1 \rangle$.

(2) $\mathfrak{a} = \langle \mathbb{Z}; \cdot, f \rangle$ where $f(z) = 2z$ (for $z \in \mathbb{Z}$).

Solution

(1) Here, $\tau^{\mathfrak{a}}[x]$ has the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ — (*) for some $n \geq 0$ and $a_0, \ldots, a_n \in \mathbb{Z}$. I.e. $\tau^{\mathfrak{a}}[x]$ is a

polynomial in x with integer coefficients.
For let F_1, F_2 be the binary function symbols of $\mathcal{L}_\sigma$ (interpreted by $+, \cdot$ respectively) and F_3 the unary function symbol (interpreted by $-$). Let c_0, c_1 be the constant symbols (interpreted by $0, 1$).

To see that $\tau^{\mathfrak{A}}[x]$ has the form $(*)$ we use induction on τ.
We first consider the case that τ is v_1, c_0 or c_1.
We have $v_1^{\mathfrak{A}}[x] = x$, $c_0^{\mathfrak{A}}[x] = 0$ and $c_1^{\mathfrak{A}}[x] = 1$ which are all of the form $(*)$ (with $n=1, a_1 = 1, a_0 = 0$; $n = 0, a_0 = 0$; $n = 0, a_0 = 1$ respectively).
Also if $\tau_1^{\mathfrak{A}}[x]$, $\tau_2^{\mathfrak{A}}[x]$ are of the form $(*)$ then $F_1(\tau_1, \tau_2)^{\mathfrak{A}}[x] = \tau_1^{\mathfrak{A}}[x] + \tau_2^{\mathfrak{A}}[x]$ can also be put into this form. Similarly for the inductive cases corresponding to F_2 and F_3.

For the converse, for $r \geq 0$, let λ_r denote the term defined inductively by

$$\lambda_0 := c_1$$
$$\lambda_{r+1} = F_2(v_1, \lambda_r).$$

Then $\lambda_r^{\mathfrak{A}}[x] = x^r$ (for all $x \in \mathbb{R}$).

Also define, $\gamma_1 := c_1$, $\gamma_{r+1} = F_1(c_1, \gamma_r)$.
Then $\gamma_r^{\mathfrak{A}}[x] = r$.

Hence $$F_2(\gamma_r, \lambda_s)^{\mathfrak{A}}[x] = r \cdot x^s, \quad \text{and}$$

$$F_3(F_2(\gamma_r, \lambda_s))^{\mathfrak{A}}[x] = -r \cdot x^s.$$

Now we obtain $(*)$ easily by induction on n, by use of F_1.

(2) Here, a similar inductive argument shows that the only functions that we can get are of the form $x \longmapsto 2^r x^s$ for some $r, s \in \mathbb{Z}$ with $r \geq 0$ and $s > 0$.
Further, all such functions are obtainable by repeated application of F_2 (multiplication) and F_1 (for f).
