Solutions 10

14.5 Exercise

Let Σ be an $\mathcal{L}$-theory and let $\phi \in F_n(\mathcal{L})$ (for some $n \geq 0$). Let $c_1, \ldots, c_n$ be constant symbols which do not occur in any sentence in Σ. Assume that $\Sigma \models \phi(c_1, \ldots, c_n)$. Then $\Sigma \models \forall v_1 \cdots \forall v_n \phi$.

Solution.

Suppose, for a contradiction, that there exists $\mathfrak{A} \models \Sigma$ such that $\mathfrak{A} \models \neg \forall v_1 \cdots \forall v_n \phi$. Then $\mathfrak{A} \models \exists v_1 \cdots \exists v_n \neg \phi$, so for some $a_1, \ldots, a_n \in A$, $\mathfrak{A} \models \neg\phi[a_1, \ldots, a_n]$. Now change the interpretation of $c_1, \ldots, c_n$ in $\mathfrak{A}$ to $a_1, \ldots, a_n$. Then we still have $\mathfrak{A}' \models \Sigma$, since $c_1, \ldots, c_n$ do not occur in any sentence in Σ, and also $\mathfrak{A}' \models \neg\phi(c_1, \ldots, c_n)$, where $\mathfrak{A}'$ is the new structure. This contradicts the fact that $\Sigma \models \phi(c_1, \ldots, c_n)$.

15.1.1 Exercise

Prove that if ϕ_1, ϕ_2 are both logically equivalent to existential sentences then so is the sentence $(\phi_1 \wedge \phi_2)$.

Solution

We may choose n sufficiently large so that ϕ_i is logically equivalent to

$$\exists v_1 \cdots \exists v_n \psi_i$$

where $\psi_i \in F_n(\mathcal{L})$ and ψ_i is QF (for $i = 1, 2$).

Then $(\phi_1 \wedge \phi_2)$ is logically equivalent to

$$\exists v_1 \cdots \exists v_n \exists v_{n+1} \cdots \exists v_{2n} (\psi_1 \wedge \psi_2')$$

where ψ_2' is the result of replacing each occurrence of v_j in ψ_2 by v_{n+j} (for $j = 1, \ldots, n$).

15.2.1 Exercise

Let $\mathfrak{A}, \mathfrak{B}$ be $\mathcal{L}$-structures with $\mathfrak{A} \subseteq \mathfrak{B}$. Consider the set of $\mathcal{L}(\mathfrak{B})$-sentences defined by

$$\Sigma := \mathrm{CDiag}(\mathfrak{A}) \cup \mathrm{Diag}(\mathfrak{B})$$

where we regard $\mathcal{L}(\mathfrak{A})$ as a sublanguage of $\mathcal{L}(\mathfrak{B})$ (i.e. we use the constant symbol c_a for each $a \in A$ in both $\mathrm{CDiag}(\mathfrak{A})$ and $\mathrm{Diag}(\mathfrak{B})$).

Now it is certainly true, as in the notes, that if $\mathfrak{C}' \models \Sigma$ and $\mathfrak{C} := \mathfrak{C}' \restriction \mathcal{L}$, then we may suppose that $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C}$ and that $\mathfrak{A} \preceq \mathfrak{C}$, which implies that

(*) for all existential $\phi \in F_n(\mathcal{L})$ and $a_1, \ldots, a_n \in A$ (and all $n \geq 0$), if $\mathfrak{B} \models \phi[a_1, \ldots, a_n]$ then $\mathfrak{A} \models \phi[a_1, \ldots, a_n]$,

and (*) is false in general. However, this argument assumes that Σ does have a model and we have no reason, a priori, to believe this. In fact:

Claim: Assume that (*) holds. Then Σ has a model (and, as we have seen, conversely).

Proof of Claim

It is sufficient to show that Σ is finitely satisfiable. Assume not. Then there is a sentence (of $\mathcal{L}(\mathfrak{B})$) $\theta \in \mathrm{Diag}(\mathfrak{B})$ such that $\mathrm{CDiag}(\mathfrak{A}) \cup \{\theta\}$ has no model. (We should really take a finite set of such sentences, but we then take the conjunction of them, which is still in $\mathrm{Diag}(\mathfrak{B})$.)

Now we may write θ in the form

$$\psi(c_{a_1}, \ldots, c_{a_n}, c_{b_1}, \ldots, c_{b_m}) \quad (\in \mathrm{Diag}(\mathcal{B})),$$

where $\psi(v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+m})$ is a QF formula of $\mathcal{L}$, $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in B \setminus A$.
Thus

$$C\mathrm{Diag}(\mathcal{A}) \models \neg\psi(c_{a_1}, \ldots, c_{a_n}, c_{b_1}, \ldots, c_{b_m}).$$

Now $c_{b_1}, \ldots, c_{b_m}$ do not occur in any sentence in $C\mathrm{Diag}(\mathcal{A})$, so by 14.5

$$C\mathrm{Diag}(\mathcal{A}) \models \forall v_{n+1} \cdots \forall v_{n+m} \neg\psi(c_{a_1}, \ldots, c_{a_n}, v_{n+1}, \ldots, v_{n+m})$$

$$\therefore \quad \mathcal{A}^+ \models \forall v_{n+1} \cdots \forall v_{n+m} \neg\psi(c_{a_1}, \ldots, c_{a_n}, v_{n+1}, \ldots, v_{n+m})$$

$$\therefore \quad \mathcal{A}^+ \models \neg\exists v_{n+1} \cdots \exists v_{n+m} \psi(c_{a_1}, \ldots, c_{a_n}, v_{n+1}, \ldots, v_{n+m})$$

$$\therefore \quad \mathcal{A} \models \neg\exists v_{n+1} \cdots \exists v_{n+m} \psi(v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+m})[a_1, \ldots, a_n]. \quad \text{--- (1)}$$

However, $\mathcal{B}^+ \models \psi(c_{a_1}, \ldots, c_{a_n}, c_{b_1}, \ldots, c_{b_m})$, so
$\mathcal{B} \models \psi(v_1, \ldots, v_{n+m})[a_1, \ldots, a_n, b_1, \ldots, b_m]$, so
$\mathcal{B} \models \exists v_{n+1} \cdots \exists v_{n+m} \psi(v_1, \ldots, v_{n+m})[a_1, \ldots, a_n]$. --- (2)

However, (1) and (2) contradict $(*)$ (with ϕ being the formula $\exists v_{n+1} \cdots \exists v_{n+m} \psi(v_1, \ldots, v_{n+m})$).
