14.5 Exercise

Let \( \Sigma \) be an \( L \)-theory and let \( \phi \in F_n(L) \) (for some \( n \geq 0 \)). Let \( c_1, \ldots, c_m \) be constant symbols which do not occur in any sentence in \( \Sigma \). Assume that \( \Sigma \models \phi(c_1, \ldots, c_m) \). Then \( \Sigma \models \forall v_1 \cdots \forall v_n \phi \).

Solution.

Suppose, for a contradiction, that there exists \( c \models \Sigma \) such that \( c \models \neg \forall v_1 \cdots \forall v_n \phi \). Then \( c \models \forall v_1 \cdots \forall v_n \neg \phi \), so for some \( a_1, \ldots, a_n \in A \), \( c \models \neg \phi[a_1, \ldots, a_n] \). Now change the interpretation of \( c_1, \ldots, c_m \) in \( c \) to \( a_1, \ldots, a_n \). Then we still have \( c \models \Sigma \), since \( c_1, \ldots, c_m \) do not occur in any sentence in \( \Sigma \), and also \( c \models \neg \phi[c_1, \ldots, c_m] \), where \( c' \) is the new structure. This contradicts the fact that \( \Sigma \models \phi(c_1, \ldots, c_m) \).

15.1.1 Exercise

Prove that if \( \phi_1, \phi_2 \) are both logically equivalent to existential sentences then so is the sentence \( (\phi_1 \land \phi_2) \).

Solution

We may choose \( n \) sufficiently large so that \( \phi_1 \) is logically equivalent to

\[ \exists v_1 \cdots \exists v_n \psi_1 \]

where \( \psi_1 \in F_n(L) \) and \( \psi \) is \( AF \) (for \( i = 1, 2 \)).

Thus \( (\phi_1 \land \phi_2) \) is logically equivalent to

\[ \exists v_1 \cdots \exists v_n \exists v_{n+1} \cdots \exists v_m (\psi_1 \land \psi_2') \]

where \( \psi_2' \) is the result of replacing each occurrence of \( v_j \) in \( \psi_2 \) by \( v_{n+j} \) (for \( j = 1, \ldots, n \)).
15.2.1 Exercise

Let $\mathfrak{C}$, $\mathfrak{D}$ be $\mathcal{L}$-structures with $\mathfrak{C} \subseteq \mathfrak{D}$. Consider the set of $\mathcal{L}(\mathfrak{D})$-sentences defined by

$$\Sigma := \text{CDrag}(\mathfrak{C}) \cup \text{Diag}(\mathfrak{D})$$

where we regard $\mathcal{L}(\mathfrak{C})$ as a sub-language of $\mathcal{L}(\mathfrak{D})$ (i.e. we use the constant symbol $c_a$ for each $a \in A$ in both CDrag($\mathfrak{C}$) and Diag($\mathfrak{D}$)).

Now it is certainly true, as in the notes, that if $\mathfrak{E}' \models \Sigma$ and $\mathfrak{E} := \mathfrak{E}' \upharpoonright \mathfrak{I}$, then we may suppose that $\mathfrak{C} \subseteq \mathfrak{E} \subseteq \mathfrak{E}'$ and that $\mathfrak{C} \subseteq \mathfrak{I}$, which implies that

(*) for all existential $\phi \in \text{E}_n(\mathfrak{E})$ and $a_1, \ldots, a_n \in A$ (and all $n \geq 0$), if $\mathfrak{E} \models \phi[a_1, \ldots, a_n]$ then $\mathfrak{E} \models \phi[a_1, \ldots, a_n]$,

and (*) is false in general. However, this argument assumes that $\Sigma$ does have a model and we have no reason, a priori, to believe this. In fact:

**Claim:** Assume that (*) holds. Then $\Sigma$ has a model (and, as we have seen, conversely).

**Proof of Claim**

It is sufficient to show that $\Sigma$ is finitely satisfiable. Assume not. Then there is a sentence $(\exists x) \phi(\bar{x})$ such that CDrag($\mathfrak{C}$) $\cup \{ \phi(\bar{x}) \}$ has no model. (We should really take a finite set of such sentences, but we then take the conjunction of them, which is still in Diag($\mathfrak{D}$)).
Now we may write $\theta$ in the form

$$\psi(c_{a_1, \ldots, c_{a_m}}, c_{b_1, \ldots, c_{b_m}}) \in \text{Diag} (\beta),$$

where $\psi(v_{i_1}, v_{i_2}, v_{i_3}, \ldots, v_{i_m})$ is a CI formula of $\mathcal{L}$, $a_1, \ldots, a_m \in A$ and $b_1, \ldots, b_m \in B \setminus A$.

Thus

$$\text{COdiag} (\alpha) = \neg \psi (c_{a_1, \ldots, c_{a_m}, c_{b_1, \ldots, c_{b_m}}).$$

Now $c_{a_1, \ldots, c_{a_m}}$ do not occur in any sentence in $\text{COdiag} (\alpha)$, so by 14.5

$$\text{COdiag} (\alpha) \models \forall v_{i_1} \ldots \forall v_{i_m} \neg \psi (c_{a_1, \ldots, c_{a_m}, v_{i_1}, \ldots, v_{i_m})$$

$$\therefore \quad \alpha^+ \models \forall v_{i_1} \ldots \forall v_{i_m} \neg \psi (c_{a_1, \ldots, c_{a_m}, v_{i_1}, \ldots, v_{i_m})$$

$$\therefore \quad \alpha^+ \models \forall v_{i_1} \ldots \forall v_{i_m} \exists v_{i_1} \ldots \exists v_{i_m} \psi (c_{a_1, \ldots, c_{a_m}, v_{i_1}, \ldots, v_{i_m})$$

$$\therefore \quad \alpha^+ \models \forall v_{i_1} \ldots \forall v_{i_m} \exists v_{i_1} \ldots \exists v_{i_m} \psi (v_{i_1}, v_{i_2}, v_{i_3}, \ldots, v_{i_m}) [a_1, \ldots, a_m] \quad \text{(1)}$$

However, $\beta^+ \models \psi (c_{a_1, \ldots, c_{a_m}, c_{b_1, \ldots, c_{b_m}})$, so

$$\beta \models \psi (v_{i_1}, v_{i_2}, v_{i_3}, \ldots, v_{i_m}) [a_1, \ldots, a_m, b_1, \ldots, b_m] \quad \text{(2)}$$

$$\beta \models \exists v_{i_1} \ldots \exists v_{i_m} \psi (v_{i_1}, v_{i_2}, v_{i_3}, \ldots, v_{i_m}) [a_1, \ldots, a_m, b_1, \ldots, b_m]$$.  

However, (1) and (2) contradict (5) (with $\phi$ being the formula $\exists v_{i_1} \ldots \exists v_{i_m} \psi (v_{i_1}, v_{i_2}, v_{i_3}, \ldots, v_{i_m})$).