Solutions 1

3.6 Exercise

Let $\mathfrak{A}, \mathfrak{A}', \mathfrak{B} \in K_\sigma$. Suppose that $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A}' \subseteq \mathfrak{B}$ and $\mathrm{dom}(\mathfrak{A}) \subseteq \mathrm{dom}(\mathfrak{A}')$. Then $\mathfrak{A} \subseteq \mathfrak{A}'$.

Solution

Writing $\mathfrak{A}, \mathfrak{B}$ as in 2.4 and $\mathfrak{A}'$ as $\langle A' ; \{R'_i\}_{i\in I} ; \{f'_j\}_{j \in J} ; \{c'_k\}_{k\in K} \rangle$, we are

given that $A \subseteq A' \subseteq B$ and (see 3.1) that
$R_i = A^n \cap S_i$ and $R'_i = (A')^n \cap S_i$ (for each $i \in I$).
Since, obviously, $A^n = (A')^n \cap A^n$ we have
$R_i = (A')^n \cap A^n \cap S_i = A^n \cap (A')^n \cap S_i = A^n \cap R'_i$. Thus 3.1(1) holds for $\mathfrak{A}, \mathfrak{A}'$.

For 3.1(2) we have $f_j = g_j \restriction A^m$, $f'_j = g_j \restriction (A')^m$ (for each $j \in J$). Again, it is clear that $f_j = f'_j \restriction A^m$ (as $A \subseteq A'$).

Finally, by 3.1(3) we have $c_k = d_k$ and $c_k' = d_k$, so $c_k = c'_k$ (for each $k \in K$).

3.8 Exercise

By inspecting the proof of 3.5, prove that if σ is countable and S is countable, then $\mathrm{dom}(\mathfrak{A})$ is countable.

Solution

Recall that $\mathrm{dom}(\mathfrak{A}) := S_\omega = \bigcup_{n=0}^{\infty} S_n$. So it is sufficient to show that each S_n is countable, because a countable union of countable sets is countable.

Now S_0 is countable as both S and K are. Suppose that S_n is countable. Then we have

$$S_{n+1} = S_n \cup \{ g_j(a_1, \dots, a_{\mu(j)}) : j \in J,\ a_1, \dots, a_{\mu(j)} \in S_n \}.$$

$$= S_n \cup \bigcup_{r=1}^{\infty} \{ g_j(a_1, \dots, a_r) : j \in J \text{ s.th. } \mu(j) = r,\ a_1, \dots, a_r \in S_n \},$$

so again it is sufficient to show that each set in the union is countable.
Now S_n is by our assumption. So consider some $r \geq 1$. Then the cardinality of $\{ g_j(a_1, \dots, a_r) : j \in J,\ \mu(j) = r,\ a_1, \dots, a_r \in S_n \}$ is no greater than that of $J \times A^r$, and this set is countable since a finite product of countable sets is countable.
Thus S_{n+1} is countable and result follows by induction.

3.9 <u>Exercise</u>

Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K_\sigma$ and suppose that $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$ and $\gamma : \mathfrak{B} \hookrightarrow \mathfrak{C}$. Prove that $\gamma \circ \pi : \mathfrak{A} \hookrightarrow \mathfrak{C}$.

<u>Solution</u>

Write $\mathfrak{A}, \mathfrak{B}$ as in 2.4 and $\mathfrak{C}$ as $\langle C ; \{T_i\}_{i \in I} ; \{h_j\}_{j \in J} ; \{b_k\}_{k \in K} \rangle$. We verify 2.4 (a)–(d).

2.4(a) Clearly $\gamma \circ \pi$ is one-one as both γ, π are. (We are not required to show $\gamma \circ \pi$ is onto – but it is if both γ and π are.)

2.4 (b). Let $i \in I$ and say $\rho(i) = n$. Then for $a_1, \dots, a_n \in A$

we have

$$\langle a_1, \dots, a_n \rangle \in R_i \text{ iff } \langle \pi(a_1), \dots, \pi(a_n) \rangle \in S_i \quad (\text{since } \pi : \mathcal{C} \hookrightarrow \mathcal{D})$$
$$\text{iff } \langle \gamma(\pi(a_1)), \dots, \gamma(\pi(a_n)) \rangle \in T_i \quad (\text{since } \gamma : \mathcal{D} \hookrightarrow \mathcal{E})$$
$$\text{iff } \langle \gamma \circ \pi(a_1), \dots, \gamma \circ \pi(a_n) \rangle \in T_i .$$

2.4 (c) Let $j \in J$, say $\mu(j) = m$. Then for $a_1, \dots, a_m \in A$ we have

$$\pi(f_j(a_1, \dots, a_m)) = g_j(\pi(a_1), \dots, \pi(a_m)) \quad (\text{since } \pi : \mathcal{C} \hookrightarrow \mathcal{D}).$$
$$\therefore \gamma(\pi(f_j(a_1, \dots, a_m))) = \gamma(g_j(\pi(a_1), \dots, \pi(a_m)))$$
$$= h_j(\gamma(\pi(a_1)), \dots, \gamma(\pi(a_m))) \quad (\text{since } \gamma : \mathcal{D} \hookrightarrow \mathcal{E})$$
$$= h_j(\gamma \circ \pi(a_1), \dots, \gamma \circ \pi(a_m)).$$

$$\therefore \gamma \circ \pi (f_j(a_1, \dots, a_m)) = h_j(\gamma \circ \pi(a_1), \dots, \gamma \circ \pi(a_m)).$$

2.4 (d) Let $k \in K$. Then $\pi(c_k) = d_k$ (since $\pi : \mathcal{C} \hookrightarrow \mathcal{D}$)

$$\therefore \gamma(\pi(c_k)) = \gamma(d_k)$$
$$\therefore \gamma \circ \pi(c_k) = b_k \quad (\text{since } \gamma : \mathcal{D} \hookrightarrow \mathcal{E}).$$