Solutions 1

3.6 Exercise

Let \( c, c', x_0 \in K \). Suppose that \( c \subseteq x_0, c' \subseteq x_0 \) and \( \text{dom}(c) \subseteq \text{dom}(c') \). Then \( c \subseteq c' \).

Solution

Writing \( c, c' \) as in 2.4 and \( c, c' \) as

\[ \langle A', \{ R_i' \}_{i \in I}, \{ f'_j \}_{j \in J}, \{ h_{i,k} \}_{i \in I, k \in K} \rangle, \]

we are given that \( A \subseteq A' \subseteq B \) and (see 3.1) that

\[ R_i = A^m \cap S_i \quad \text{and} \quad R_i' = (A')^m \cap S_i \quad (\text{for each } i \in I). \]

Since, obviously, \( A^m = (A')^m \cap A^m \) we have

\[ R_i = (A')^m \cap A^m \cap S_i = A^m \cap (A')^m \cap S_i = A^m \cap R_i'. \]

Thus 3.1 (1) holds for \( c, c' \).

For 3.1 (2) we have

\[ f_j = g_j \cap A^m, \quad f'_j = g_j \cap (A')^m \quad (\text{for each } j \in J). \]

Again, it is clear that

\[ f'_j = f'_j \cap A^m \quad (\text{for } j \in J). \]

Finally, by 3.1 (3) we have

\[ e_k = d_k \quad \text{and} \quad e'_k = d_k, \quad (\text{for each } k \in K). \]

3.8 Exercise

By inspecting the proof of 3.5, prove that if \( \sigma \) is countable and \( S \) is countable, then \( \text{dom}(\sigma) \) is countable.

Solution

Recall that \( \text{dom}(\sigma) = S_\infty = \bigcup_{n=-\infty}^{\infty} S_n. \) So it is sufficient to show that each \( S_n \) is countable, because a countable union of countable sets is countable.
Now $S_0$ is countable as both $S$ and $K$ are.

Suppose that $S_n$ is countable. Then we have

$$S_{n+1} = S_n \cup \bigcup_{r=1}^{\infty} \{ q_r(a_1, \ldots, a_n, j) : j \in J, a_1, \ldots, a_n \in S_n \}.$$ 

so again it is sufficient to show that each set in the union is countable.

Now $S_n$ is by our assumption. So consider some $r \geq 1$.

Then the cardinality of $\{ q_r(a_1, \ldots, a_n) : j \in J, \mu(j) = r, a_1, \ldots, a_n \in S_n \}$ is no greater than that of $J \times A^r$, and this set is countable since a finite product of countable sets is countable.

Thus $S_{n+1}$ is countable and result follows by induction.

3.9 Exercise

Let $X$, $Y$, $Z$ be sets and suppose that $\pi : X \to Y$ and $\gamma : Y \to Z$. Prove that $\gamma \circ \pi : X \to Z$.

Solution

Write $X$, $Y$, $Z$ as in 2.4 and $E$ as

$$\langle E; E_{i \in I}; \{ E_{i,j} \}_{i,j \in J}; \{ E_{k,l} \}_{k,l \in I} \rangle.$$ We verify 2.4(a) - (d).

2.4(a) Clearly $\gamma \circ \pi$ is one-one as both $\gamma$, $\pi$ are. (We are not required to show $\gamma \circ \pi$ in onto - but it is if both $\gamma$ and $\pi$ are.)

2.4(b). Let $i \in I$ and say $\rho(i) = n$. Then for $a_1, \ldots, a_n \in A$
we have
\[ \langle a_1, \ldots, a_n \rangle \in R_i \quad \text{iff} \quad \langle \pi(a_1), \ldots, \pi(a_n) \rangle \in S_i \quad (\text{since } \pi : C \rightarrow \mathbb{C}) \]
\[ \text{iff } \langle \gamma(\pi(a_1)), \ldots, \gamma(\pi(a_n)) \rangle \in T_i \quad (\text{since } \gamma : \mathbb{C} \rightarrow \mathbb{C}) \]
\[ \text{iff } \langle \gamma_0 \pi(a_1), \ldots, \gamma_0 \pi(a_n) \rangle \in T_i . \]

2.4 (c) Let \( j \in J \), say \( \mu(j) = m \). Then for \( a_1, \ldots, a_m \in A \) we have
\[ \pi(\xi_j(a_1, \ldots, a_m)) = \gamma_j(\pi(a_1), \ldots, \pi(a_m)) \quad (\text{since } \pi : C \rightarrow \mathbb{C}) . \]
\[ \gamma(\pi(\xi_j(a_1, \ldots, a_m))) = \gamma(\xi_j(\pi(a_1), \ldots, \pi(a_m))) \]
\[ = \xi_j(\gamma(\pi(a_1)), \ldots, \gamma(\pi(a_m))) \quad (\text{since } \gamma : \mathbb{C} \rightarrow \mathbb{C}) . \]
\[ = \xi_j(\gamma_0 \pi(a_1), \ldots, \gamma_0 \pi(a_m)) . \]
\[ \therefore \gamma_0 \pi(\xi_j(a_1, \ldots, a_m)) = \xi_j(\gamma_0 \pi(a_1), \ldots, \gamma_0 \pi(a_m)) . \]

2.4 (d) Let \( k \in K \). Then \( \pi(l_k) = d_k \quad (\text{since } \pi : C \rightarrow \mathbb{C}) \)
\[ \gamma(\pi(l_k)) = \gamma(d_k) \]
\[ \therefore \gamma_0 \pi(l_k) = d_k \quad (\text{since } \gamma : \mathbb{C} \rightarrow \mathbb{C}) . \]