13.10 <u>Exercise</u>

Let $\mathcal{L}$ be any language and let $\phi(v_1, \ldots, v_n) \in F_n(\mathcal{L})$. Let $\gamma_1, \ldots, \gamma_n$ be constant symbols <u>not</u> in $\mathcal{L}$. (i) Assume that $\Sigma$ is an $\mathcal{L}$-theory and that $\Sigma \models \phi(\gamma_1, \ldots, \gamma_n)$. Then $\Sigma \models \forall v_1 \ldots \forall v_n \phi(v_1, \ldots, v_n)$.
(ii) Now assume that $\Sigma$ is an $\mathcal{L}$-theory, that $\psi$ is an $\mathcal{L}$-sentence and that $\Sigma \cup \{\phi(\gamma_1, \ldots, \gamma_n)\} \models \psi$. Then $\Sigma \cup \{\exists v_1 \ldots \exists v_n \phi(v_1, \ldots, v_n)\} \models \psi$.

13.11 <u>Proof of the Omitting Types Theorem (13.8)</u>

Let $p \in S_n(T)$ be a <u>non-principal</u> type where $T$ is a complete $\mathcal{L}$-theory in a countable language $\mathcal{L}$. We must find a countable model $\mathfrak{A} \models T$ such that $\mathfrak{A}$ omits $p$. We shall construct $\mathfrak{A}$ as a <u>term model</u> for some extended language $\mathcal{L}'$ $(\supseteq \mathcal{L})$. This means that for every $a \in A$, there is some closed term $\tau$ of $\mathcal{L}'$ such that $\tau^{\mathfrak{A}} = a$. This is just as in the proof of the Compactness Theorem and, indeed, the construction has many of the same steps. There is one extra step, however, (the step guaranteeing that $p$ is omitted in $\mathfrak{A}$) and the countability of $\mathcal{L}$ is needed at this point.

So let $c_1, c_2, \ldots, c_m, \ldots$ be a countably infinite sequence of new constant symbols. Add these to $\mathcal{L}$ to get the extended language $\mathcal{L}'$.
Now let $\phi_0, \phi_1, \ldots, \phi_m, \ldots$ be an enumeration of all sentences of $\mathcal{L}'$ and let $\alpha_0, \alpha_1, \ldots, \alpha_m, \ldots$ be an enumeration of all $n$-tuples, $\langle c_{j_1}, \ldots, c_{j_n} \rangle$ of the new constant symbols.
We construct, for each $m \geq 0$, a finite set $S_m$ of $\mathcal{L}'$-sentences such that $T \cup S_m$ is satisfiable as follows.
Set $S_0 = \emptyset$.

Now suppose that $S_m$ has been constructed for some $m \geq 0$.

<u>Case 1</u> $m$ is odd, say $m = 2l+1$ for some $l \geq 0$.
Let $\mathfrak{A}'$ be any $\mathcal{L}'$-structure such that $\mathfrak{A}' \models T \cup S_m$. Consider the sentence $\phi_l$. If $\mathfrak{A}' \models \phi_l$, let $S_{m+1} := S_m \cup \{\phi_l\}$. If $\mathfrak{A}' \models \neg\phi_l$, let $S_{m+1} := S_m \cup \{\neg\phi_l\}$. Clearly $T \cup S_{m+1}$ is satisfiable (by $\mathfrak{A}'$).

<u>Case 2</u> $m$ is even, but not divisible by 4. Say $m = 2\cdot(2l+1)$.
Suppose $\phi_l$ has the form $\exists v_j \psi(v_j)$ and $T \cup S_m \models \exists v_j \psi(v_j)$ (If one of these two conditions fails, just let $S_{m+1} := S_m$.)
Now choose $k \geq 1$ so that the constant symbols occurring in $S_m \cup \{\exists v_j \psi(v_j)\}$ are amongst $\{c_1, \ldots, c_{k-1}\}$ and let $\mathcal{L}'_k$ be the language $\mathcal{L}$ together with just $c_1, \ldots, c_{k-1}$. I claim that $T \cup S_m \cup \{\psi(c_k)\}$ is satisfiable. For if not, $T \cup S_m \models \neg\psi(c_k)$ and hence $T \cup S_m \models \forall v_j \neg\psi(v_j)$ by 13.10 (i) (applied to $\mathcal{L} = \mathcal{L}'_k$). Thus $T \cup S_m \models \neg\exists v_j \psi(v_j)$. But since $T \cup S_m \models \exists v_j \psi(v_j)$ it follows that $T \cup S_m$ is not satisfiable – contradiction. Thus, we may set $S_{m+1} := S_m \cup \{\psi(c_k)\}$.

<u>Case 3</u> $m$ is divisible by 4, say $m = 4l$.
This is the new case. We want to construct $S_{m+1}$ so that if $\mathfrak{A}'$ is <u>any</u> model of $T \cup S_{m+1}$, then the $n$-tuple $\langle c_{j_1}^{\mathfrak{A}'}, \ldots, c_{j_n}^{\mathfrak{A}'} \rangle \in A'^n$ does not realise $p$, where $\langle c_{j_1}, \ldots, c_{j_n} \rangle$ is $\alpha_l$.
This will clearly be achieved if we can find a formula $\phi(v_1, \ldots, v_n) \in p$ such that $T \cup S_m \cup \{\neg\phi(c_{j_1}, \ldots, c_{j_n})\}$ is satisfiable, for then we set $S_{m+1} := S_m \cup \{\neg\phi(c_{j_1}, \ldots, c_{j_n})\}$.
Suppose, for a contradiction, that there is no such formula $\phi$.

Then for *every* formula $\phi(v_1, \dots, v_n) \in p$, we have that $T \cup S_m \cup \{\neg \phi(c_{j_1}, \dots, c_{j_n})\}$ is *not* satisfiable. So for *every* $\phi(v_1, \dots, v_n) \in p$, $T \cup S_m \models \phi(c_{j_1}, \dots, c_{j_n})$ ----- $(*)$.

Now let $\chi'$ be the conjunction of the (finitely many) sentences in $S_m$. Then, up to logical equivalence, we may write $\chi'$ in the form $\chi(c_{j_1}, \dots, c_{j_n}, c_{i_1}, \dots, c_{i_s})$ where $\chi(v_1, \dots, v_n, v_{n+1}, \dots, v_{n+s})$ is a formula of $\mathcal{L}$. By $(*)$ we have

$T \cup \{\chi(c_{j_1}, \dots, c_{j_n}, c_{i_1}, \dots, c_{i_s})\} \models \phi(c_{j_1}, \dots, c_{j_n})$ for *all* $\phi \in p$.

(Here, $c_{i_1}, \dots, c_{i_s}$ are the new constant symbols occurring in $\chi'$ that are not amongst $c_{j_1}, \dots, c_{j_n}$.)

We now apply 13.10 (ii) with $\mathcal{L}$ there equal to $\mathcal{L} \cup \{c_{j_1}, \dots, c_{j_n}\}$ here, to get

$T \cup \{\exists v_{n+1} \cdots \exists v_{n+s}\, \chi(c_{j_1}, \dots, c_{j_n}, v_{n+1}, \dots, v_{n+s})\} \models \phi(c_{j_1}, \dots, c_{j_n})$ for *all* $\phi \in p$.

Thus $T \models (\theta(c_{j_1}, \dots, c_{j_n}) \to \phi(c_{j_1}, \dots, c_{j_n}))$ for *all* $\phi \in p$.

But since $c_{j_1}, \dots, c_{j_n}$ do not occur in $T$, we may apply 13.10(i) (to our original $\mathcal{L}$) and obtain

$(\dagger) \dots \quad T \models \forall v_1 \cdots \forall v_n \, (\theta(v_1, \dots, v_n) \to \phi(v_1, \dots, v_n))$ for *all* $\phi \in p$.

However, $\theta(v_1, \dots, v_n) \in p$ (because otherwise $\neg\theta(v_1, \dots, v_n) \in p$, and taking $\phi$ to be $\neg\theta$ in $(\dagger)$ we see that $T \models \forall v_1 \dots \forall v_n \neg\theta(v_1, \dots, v_n)$ which easily implies that $T \cup S_m$ is not satisfiable), so by $(\dagger)$ we see that $p$ is a principal $n$type (over $T$) – contradiction.

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This completes the inductive construction of the $S_m$'s and we now set $\Sigma' := T \cup \bigcup_{m \geq 0} S_m$.

Then $\Sigma'$ is finitely satisfiable since if $\Sigma_0 \subseteq_{\text{finite}} \Sigma'$ then $\Sigma_0 \subseteq T \cup S_{m_0}$ for some $m_0$ (NB $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_m \subseteq \cdots$) and we guaranteed that $T \cup S_{m_0}$ was satisfiable.

Also, by cases (i) and (ii) of the construction $\Sigma'$ is both complete and full (in the language $\mathcal{L}'$). Hence, by exactly the same proof as in that of the Compactness Theorem, $\Sigma'$ has a model $\mathfrak{A}'$ say, such that $A' = \{\tilde{\tau} : \tau \text{ a term of } \mathcal{L}'\}$. In fact, since for any closed term $\tau$, $\models \exists v_1\, v_1 \simeq \tau$, it follows from case (ii) of the construction that $\Sigma' \models c_k \simeq \tau$ for some new constant symbol $c_k$. Hence $c_k \in \tilde{\tau}$, so we actually have $A' = \{\tilde{c}_i : i = 1, 2, \dots\}$. Thus $\mathfrak{A}' = \langle \mathfrak{A}; \tilde{c}_1, \tilde{c}_2, \dots \rangle$ where $\mathfrak{A}$ is an $\mathcal{L}$-structure such that $\mathfrak{A} \models T$. Notice that $\text{dom}(\mathfrak{A}) = \{\tilde{c}_i : i = 1, 2, \dots\}$ is countable. We complete the proof by showing that $\mathfrak{A}$ omits $p$.

So suppose, for a contradiction, that $\mathfrak{A}$ realises $p$. Then there exists $\langle \tilde{c}_{j_1}, \dots, \tilde{c}_{j_n} \rangle \in A'^n$ such that $\mathfrak{A} \models \phi[\tilde{c}_{j_1}, \dots, \tilde{c}_{j_n}]$ for all $\phi \in p$. Now by the construction of the term model (see proof of Compactness Theorem), this means that $\phi(c_{j_1}, \dots, c_{j_n}) \in \Sigma'$ for all $\phi \in p$. However, choosing $l$ so that $\alpha_l = \langle c_{j_1}, \dots, c_{j_n} \rangle$ we see from ~~stage~~ case (iii) of the construction that for some $\phi \in p$, $\neg\phi(c_{j_1}, \dots, c_{j_n}) \in S_{4l+1}$, so $\neg\phi(c_{j_1}, \dots, c_{j_n}) \in \Sigma'$ – contradicting the satisfiability of $\Sigma'$. $\square$

<u>Exercise</u>

Modify the above construction to show that if $\{p_i : i \geq 1\}$ is any countable collection of non-principal types (over $T$) (possibly having different $n$'s) then there exists a countable $\mathfrak{A} \models T$ which simultaneously omits all the $p_i$.