Exercise

Let $L$ be any language and let $\phi(x_1, \ldots, x_n) \in F_n(L)$. Let $\gamma_1, \ldots, \gamma_n$ be constant symbols not in $L$. (i) Assume that $\Sigma$ is an $L$-theory and that $\Sigma \models \phi(\gamma_1, \ldots, \gamma_n)$. Then $\Sigma \models \forall x_1 \ldots \forall x_n \phi(x_1, \ldots, x_n)$.

(ii) Now assume that $\Sigma$ is an $L$-theory, that $\psi$ is an $L$-sentence and that $\Sigma \cup \{ \phi(\gamma_1, \ldots, \gamma_n) \} \models \psi$. Then $\Sigma \cup \{ \forall x_1 \ldots \forall x_n \phi(x_1, \ldots, x_n) \} \models \psi$.

13.11 Proof of the Omitting Types Theorem (13.8)

Let $p \in S_m(T)$ be a non-principal type where $T$ is a complete $L$-theory in a countable language $L$. We must find a countable model $M \models T$ such that $M$ omits $p$. We shall construct $M$ as a term model for some extended language $L' (\supset L)$. This means that for every $a \in A$, there is some relored term $t$ of $L'$ such that $M \models t = a$. This is just as in the proof of the Compactness Theorem, and, indeed, the construction has many of the same steps. There is one extra step, however, (the step guaranteeing that $p$ is omitted in $M$) and the countability of $L$ is needed at this point.

So let $c_1, c_2, \ldots, c_m, \ldots$ be a countably infinite sequence of new constant symbols. Adjoin them to $L$ to get the extended language $L'$.

Now let $\phi_1, \phi_2, \ldots, \phi_m, \ldots$ be an enumeration of all sentences of $L'$ and let $d_1, d_2, \ldots, d_m, \ldots$ be an enumeration of all $n$-tuples, $<c_1, \ldots, c_n>$ of the new constant symbols.

We reconstruct, for each $m \geq 0$, a finite set $S_m$ of $L'$-sentences such that $T \cup S_m$ is satisfiable as follows.

Set $S_0 = \emptyset$. 
Now suppose that $S_m$ has been constructed for some $m \geq 0$.

Case 1. $m$ is odd, say $m = 2l + 1$ for some $l \geq 0$.

Let $C_1'$ be any $L'$-structure such that $C_1' \models T_0 S_m$. Consider the sentence $\phi_3$. If $C_1' \models \phi_3$, let $S_{m+1} := S_m \cup \{ \phi_3 \}$. If $C_1' \not\models \phi_3$, let $S_{m+1} := S_m \cup \{ \neg \phi_3 \}$. Clearly, $T_0 S_{m+1}$ is satisfiable (by $C_1'$).

Case 2. $m$ is even, but not divisible by 4. Say $m = 2(2l + 1)$.

Suppose $\phi_3$ has the form $\exists v_j \psi(v_j)$ and $T_0 S_m \models \exists v_j \psi(v_j)$. If one of these two conditions fails, just let $S_{m+1} := S_m$.

Now choose $k \geq 1$ so that the constant symbols occurring in the sentence $S_m \cup \exists v_j \psi(v_j)$ are amongst $\{c_1, \ldots, c_k \}$ and let $L_k$ be the language $L$ together with just $c_1, \ldots, c_k$. I claim that this language $L_k$ is satisfiable. For if not, $T_0 S_m \models \neg \psi(c_k)$ and hence $T_0 S_m \models \forall v_j \neg \psi(v_j)$ by 13.10(i) (applied to $L = L_k$), and therefore $T_0 S_m \models \neg \exists v_j \psi(v_j)$. But since $T_0 S_m \models \exists v_j \psi(v_j)$, it follows that $T_0 S_m$ is not satisfiable — contradiction. Thus, we may set $S_{m+1} := S_m \cup \{ \psi(c_k) \}$.

Case 3. $m$ is divisible by 4, say $m = 4l$.

This is the new case. We want to construct $S_{m+1}$ so that if $C_1'$ is any model of $T_0 S_{m+1}$, then the $\alpha$-tuple $< c_{j_1}, \ldots, c_{j_n}, S_m > \in A^m$ does not realize $\phi$, where $< c_{j_1}, \ldots, c_{j_n} >$ is in $A^m$.

This will clearly be achieved if we can find a formula $\phi(v_1, \ldots, v_n) \in \overline{P}$ such that $T_0 S_m \cup \{ \neg \phi(c_{j_1}, \ldots, c_{j_n}) \}$ is satisfiable, for then we let $S_{m+1} := S_m \cup \{ \neg \phi(c_{j_1}, \ldots, c_{j_n}) \}$.

Suppose, for a contradiction, that there is no such formula $\phi$. 

Then for every formula $\phi(v_1, \ldots, v_n) \in P$, we have that $TuS_m \cup \{\neg \phi(c_j, \ldots, c_{j+1})\}$ is not satisfiable. So for every $\phi(v_1, \ldots, v_n) \in P$, $TuSm \not\models \phi(c_{j}, \ldots, c_{j+1})$ --- (x).

Now let $X'$ be the conjunction of the (finitely many) sentences in $Sm$. Then, up to logical equivalence, we may write $X'$ in the form $X(c_{j}, \ldots, c_{j+n}, c_i, \ldots, c_{i+\delta})$ where $X(v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+\delta})$ is a formula of $L$. By (x) we have $Tu\not\models \{X(c_{j}, \ldots, c_{j+n}, c_i, \ldots, c_{i+\delta})\} \not\models \phi(c_{j}, \ldots, c_{j+1})$ for all $\phi \in P$.

(Here, $c_i, \ldots, c_{i+\delta}$ are the new constant symbols occurring in $X'$ that are not amongst $c_j, \ldots, c_{j+n}$.)

We now apply 13.10(ii) with $L$ there equal to $L u\{c_j, \ldots, c_{j+n}\}$ here, to get $Tu\not\models \exists_{n+1} \ldots \exists_{n+\delta} X(c_{j}, \ldots, c_{j+n}, v_{n+1}, \ldots, v_{n+\delta}) \not\models \phi(c_{j}, \ldots, c_{j+1})$ for all $\phi \in P$.

Thus $T = (\Theta(c_{j}, \ldots, c_{j+n}) \rightarrow \phi(c_{j}, \ldots, c_{j+n}))$ for all $\phi \in P$.

But since $c_i, \ldots, c_{i+\delta}$ do not occur in $T$, we may apply 13.10(i) (to our original $L$) and obtain

$$Tu \not\models \forall v_1 \ldots \forall v_n (\Theta(v_1, \ldots, v_n) \rightarrow \phi(v_1, \ldots, v_n))$$ for all $\phi \in P$.

However, $\Theta(v_1, \ldots, v_n) \in P$ (because otherwise $\not\models \theta(v_1, \ldots, v_n) \in P$, and taking $\phi$ to be $\neg \theta$ in (4) we see that $T \not\models \forall v_1 \ldots \forall v_n \neg \Theta(v_1, v_n)$) which clearly implies that $TuS_m$ is not satisfiable), so by (4) we see that $P$ is a principal n-type (over $T$) --- CONTRADICTION.

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This completes the inductive construction of the $S_m$'s and we now set $\Sigma' := Tu_{m \geq 0} S_m$. 
Then $\Sigma'$ is finitely satisfiable since if $\Sigma_0 \subseteq \Sigma'$, then $\Sigma_0 \subseteq T \cup S_m$, for some $m_0$. (NB: $\Sigma_0 \subseteq S_1 \subseteq \ldots \subseteq S_m \subseteq \ldots$) and we guaranteed that $T \cup S_m$ was satisfiable.

Also, by cases (i) and (ii) of the construction $\Sigma'$ is both complete and full (in the language $L'$). Hence, by exactly the same proof as in that of the Compactness Theorem, $\Sigma'$ has a model $C$, say, such that $A' = \{ \bar{e} : \bar{e} \text{ a term of } L' \}$. In fact, since for any closed term $\tau$, $\Sigma' \models \tau = \bar{c}$, it follows from case (ii) of the construction that $\Sigma' \models \bar{e}_k = \bar{c}$ for some new constant symbol $\bar{c}_k$. Hence $\bar{c}_k \in \Sigma'$, so we actually have $A' = \{ \bar{e}_i : i = 1, 2, \ldots \}$. Thus $C' = \langle C, \bar{c}_1, \bar{c}_2, \ldots \rangle$.

where $C$ is an $L$-structure such that $C \models T$. Notice that dom$(C) = \{ \bar{e}_i : i = 1, 2, \ldots \}$ is countable. We complete the proof by showing that $C$ omits $p$.

So suppose, for a contradiction, that $C$ realizes $p$. Then there exist $\langle \bar{e}_0, \ldots, \bar{e}_n \rangle \in A'$ such that $C \models \phi[\bar{e}_0, \ldots, \bar{e}_n]$ for all $\phi \in p$. Now by the construction of the term model (see proof of Compactness Theorem), this means that $\phi(\bar{c}_0, \ldots, \bar{c}_n) \in \Sigma'$ for all $\phi \in p$. However, choosing $\Gamma$ so that $\Delta_\phi = \langle \bar{c}_0, \ldots, \bar{c}_n \rangle$, we see from case (iii) of the construction that for some $\phi \in p$,

$\neg \phi(\bar{c}_0, \ldots, \bar{c}_n) \in S_{4k+1}$, so $\neg \phi(\bar{c}_0, \ldots, \bar{c}_n) \in \Sigma'$. Contradicting the satisfiability of $\Sigma$.

Exercise

Modify the above construction to show that if $\{ p_i : i \geq 1 \}$ is any countable collection of non-principal types (over $T$) (possibly having different $n$’s) then there exists a countable $C \models T$ which simultaneously omits all the $p_i$.  

$\square$