Three hours

THE UNIVERSITY OF MANCHESTER

FOURIER ANALYSIS AND LEBESGUE INTEGRATION

24th January 2013

9.45 – 12.45

Answer ALL SIX questions in Section A (25 marks in total).
Answer THREE of the FOUR questions in Section B (75 marks in total).
Answer BOTH questions in Section C (50 marks in total).

If more than THREE questions from Section B are attempted
then credit will be given for the BEST THREE answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** six questions

A1. Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with period 2π and such that both f and f' are piecewise continuous.

(a) Write down the *Fourier Series* associated to f, giving formulae for the coefficients.

(b) Write down a formula for the value of the Fourier series associated to f at a point where f is *not* continuous.

[4 marks]

A2. Write down which of the following sets are countable:

$$[0,1], \quad \mathbb{N} \times \mathbb{N}, \quad \mathbb{Q}, \quad \mathbb{R}\,.$$

[2 marks]

A3. (i) What is meant by saying that a collection $\mathcal{A}$ of subsets of a set X is a σ-algebra?

(ii) Explain how the Borel σ-algebra of subsets of $\mathbb{R}$ is defined. Show that the Borel σ-algebra contains all *countable* subsets of $\mathbb{R}$.

(iii) Give an example of a σ-algebra of subsets of $[0,1]$ which contains all countable subsets of $[0,1]$ but does not contain all Borel subsets of $[0,1]$. (You do not need to prove that your example is a σ-algebra.)

[6 marks]

A4. (i) Give a definition of $\mathcal{M}$, the collection of measurable sets.

(ii) What is meant by saying that $\mu : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ satisfies
  (a) the length property,
  (b) translation invariance?

[4 marks]

A5. What is meant by saying that $f : [0,1] \to \mathbb{R}$ is simple? For a simple function f, how is $\int f d\mu$ defined?

[4 marks]

A6.
(i) What does it mean for a measurable function $f : [-\pi, \pi] \to \mathbb{R}$ to be square integrable?

(ii) Give the definition of the space $L^2([-\pi,\pi], \mu, \mathbb{R})$ and the metric associated with it.

[5 marks]

SECTION B

Answer **THREE** of the four questions

B7. (i) State what it means for $E \subset \mathbb{R}$ to be a null set.

[2 marks]

(ii) Show that a countable set is a null set.

[8 marks]

Let D be the set obtained by the following modification of the Middle Third Cantor set. Let $D_0 = [0,1]$. Obtain a new set D_1 by removing the open middle half $\left(\frac{1}{4}, \frac{3}{4}\right)$, of D_0, so that $D_1 = \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$. Repeat this procedure to obtain sets $D_2, D_3, D_4, \ldots$ and define

$$D = \bigcap_{n=0}^{\infty} D_n \,.$$

(iii) How many disjoint intervals make up D_n? What is their length?

[2 marks]

(iv) Write down (without proof) which of the following properties D has:

(a) D is a null set.
(b) D is a Borel set.
(c) D is a measurable set.
(d) D is an open set.

[4 marks]

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(v) Show that D is uncountable. (You may assume that each point in D is of the form $\sum_{i=1}^{\infty} a_i 4^{-i}$ with $a_i \in \{0,3\}$ for $i \in \mathbb{N}$, but you may not assume without proof that this representation is unique. You may assume that $\{0,1\}^{\mathbb{N}}$ is uncountable.)

[9 marks]

B8. (i) What is meant by saying that a function $f : [0, 1] \to \mathbb{R}^*$ is measurable?

[3 marks]

(ii) Let $f : [0, 1] \to \mathbb{R}$ be a measurable function. Show that the functions $g_1, g_2 : [0, 1] \to \mathbb{R}$ defined by

$$g_1(x) = \text{ the greatest integer less or equal to } f(x),$$
$$g_2(x) = f(x) - g_1(x)$$

are measurable. You may use any convenient criteria of measurability of functions, and results about measurable functions from the course.

[8 marks]

(iii) Let $f_n : [0, 1] \to \mathbb{R}^*$, $n \geq 1$, be a sequence of measurable functions. Show that $\limsup_{n\to+\infty} f_n$ is measurable. (You may find it useful to show first that $\sup_n f_n$ and $\inf_n f_n$ are measurable.)

[9 marks]

(iv) What is meant by saying that two function $f, g : [0, 1] \to \mathbb{R}^*$ are equal μ-*almost everywhere*? Give an example of a function $f : [0, 1] \to \mathbb{R}^*$ which is equal to a *continuous* function $g : [0, 1] \to \mathbb{R}^*$ μ-almost ewerywhere but such that f is discontinuous at infinitely many points.

[5 marks]

B9. (i) (a) How is $\int f d\mu$ defined when $f : [0,1] \to \mathbb{R}^*$ is a *nonnegative* measurable function?

(b) What is meant by saying that a general measurable function $f : [0,1] \to \mathbb{R}^*$ is an *integrable* function? How is $\int f d\mu$ defined?

[4 marks]

(ii) State (without proof) the Monotone Convergence Theorem. Use it to prove the following special case of Fatou's Lemma: if f_n, $n \geq 1$ is a sequence of nonnegative integrable functions then

$$\int \liminf_{n\to+\infty} f_n \, d\mu \leq \liminf_{n\to+\infty} \int f_n \, d\mu \, .$$

[8 marks]

(iii) State (without proof) the Dominated Convergence Theorem. Use it to find $\lim_{n\to\infty} \int f_n \, d\mu$ when $f_n : [0,1] \to \mathbb{R}$ are defined by

$$f_n(x) = \frac{1 + nx^2}{(1 + x^2)^n} \, .$$

(You may use the facts that for a real number $a > 0$, $1 + na \leq (1 + a)^n$ and $\lim_{n\to\infty} \frac{n}{(1+a)^n} = 0$.)

[9 marks]

(iv) Give an example of a sequence of integrable functions $f_n : [0,1] \to \mathbb{R}$ which converge pointwise to a function $f : [0,1] \to \mathbb{R}$ but such that $\int f_n \, d\mu$ does not converge to $\int f d\mu$.

[4 marks]

B10. (i) Let g_i for $i = 1, 2, \ldots$ be functions from $L^2([-\pi,\pi],\mu,\mathbb{R})$ such that for all i

$$\| g_{i+1} - g_i \|_2 \leq \frac{1}{2^i} \, .$$

Let $g_0 : [-\pi,\pi] \to \mathbb{R}$ be the constant function equal to 0 and define $h_n, h : [-\pi,\pi] \to \mathbb{R}^*$ (for $n \geq 1$) by

$$h_n(x) = \sum_{i=0}^{n-1} |g_{i+1}(x) - g_i(x)| \, ,$$
$$h(x) = \lim_{n\to\infty} h_n(x) \quad .$$

(a) Using the Monotone Convergence Theorem, show that $h \in L^2([-\pi,\pi],\mu,\mathbb{R})$.

[6 marks]

(b) Noting that for $n \geq 1$,

$$g_n(x) = \sum_{i=0}^{n-1} (g_{i+1}(x) - g_i(x)),$$

deduce that the sequence $g_n(x)$, $n \geq 1$ converges (to a value in $\mathbb{R}$) for μ-a.e. x.

[4 marks]

(c) Define

$$g(x) = \begin{cases} \lim_{n\to\infty} g_n(x) & \text{if the limit (in } \mathbb{R}\text{) exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Show that $g \in L^2([-\pi,\pi],\mu,\mathbb{R})$ and that $\lim_{n\to\infty} \|g_n - g\|_2 = 0$.

[6 marks]

(d) Starting with an arbitrary Cauchy sequence f_n, $n \geq 1$ of functions in $L^2([-\pi,\pi],\mu,\mathbb{R})$, use the above with suitably chosen $g_i = f_{N_i}$ to conclude that $L^2([-\pi,\pi],\mu,\mathbb{R})$ is complete.

[6 marks]

(ii) State (without proof) the Riesz-Fisher Theorem.

[3 marks]

SECTION C

Answer **BOTH** questions

C11. (i) Show that for any $E \subseteq \mathbb{R}$

$$\mu^*(E) = \inf\left\{\sum_j l(I_j) : \{I_j\} \text{ is a cover of E by intervals}\right\}$$

is equal to

$$\mu_o^*(E) = \inf\left\{\sum_j l(I_j)\ :\ \{I_j\} \text{ is a cover of E by open intervals}\right\}.$$

[4 marks]

(ii) Prove that the Lebesgue Outer Measure μ^* satisfies the property of *Regularity*:

For all $E \subset \mathbb{R}$, $\mu^*(E) = \inf\{\mu^*(U)\ :\ U \text{ is open and } E \subset U\}$.

[4 marks]

In what follows, $\mathcal{M}_0$ is the collection of sets $A \subset \mathbb{R}$ such that for every set $X \subset \mathbb{R}$,

$$\mu^*(A \cap X) + \mu^*(A^c \cap X) = \mu^*(X).$$

(iii) Show that μ^* restricted to $\mathcal{M}_0$ satisfies the countable additivity, that is, for pairwise disjoint sets $A_1, A_2, \ldots$ from $\mathcal{M}_0$,

$$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu^*(A_j).$$

You may assume that μ^* satisfies countable subadditivity.

[10 marks]

(iv) Assuming that $\mathcal{M}_0$ is a σ-algebra and that it contains all intervals (a, ∞) for $a \in \mathbb{R}$, prove that all Borel sets are contained in $\mathcal{M}_0$.

[7 marks]

C12. You may assume properties of μ and μ^* proved in the course.

(i) Assume that $\{E_j\}_{j=1}^{\infty}$ is a countable collection of pairwise disjoint subsets of $[0,1)$ such that

$$\bigcup_{j=1}^{\infty} E_j = [0,1) \quad \text{and} \quad \mu^*(E_j) = \mu^*(E_k) \;\; \text{for all } j,k \in \mathbb{N}\,.$$

(a) Show that the common value of $\mu^*(E_j)$ is greater than 0.

(b) Show that only finitely many of the E_j can be measurable.

[7 marks]

(ii) Assuming the existence a non-measurable set, show that there are disjoint sets $A, B \subset \mathbb{R}$ such that $\mu^*(A) + \mu^*(B) \neq \mu^*(A \cup B)$.

[3 marks]

(iii) For any $E \subset \mathbb{R}$ and $r > 0$ define $rE = \{rx : x \in E\}$.

(a) Show that $\mu^*(rE) = r\mu^*(E)$.

(b) Show that rE is measurable if and only if E is measurable.

(c) Deduce that for any $a > 0$ there is a non-measurable set with $\mu^*(A) = a$.
(You may assume the existence of a non-measurable set E with $\mu^*(E) > 0$.)

[9 marks]

(iv) (a) Let X be a set and $\mathcal{A}$ a σ-algebra of subsets of X. Define what it means for a function $m : \mathcal{A} \to \mathbb{R}^+ \cup \{\infty\}$ to be a measure.

(b) Let $X = [0,1]$ and $\mathcal{A} = \mathcal{M}([0,1])$, and let $f : [0,1] \to \mathbb{R}$ be a nonnegative measurable function. Show that $m : \mathcal{A} \to \mathbb{R}^+ \cup \{+\infty\}$ defined by

$$m(A) = \int f \chi_A d\mu$$

is a measure.

[6 marks]

END OF EXAMINATION PAPER