MODEL THEORY MATH 43052/63052

Solutions and mark scheme

Question 1

(a) (i) The only terms are variables.

(ii) So the only atomic formulas are of the form $v_i \doteq v_j$ or $P(v_i, v_j)$.

(iii) • Every atomic formula is a formula;

• If ϕ, ψ are formulas, so are $\neg\phi$, $(\phi \wedge \psi)$, $\exists v_i \phi$.

Bookwork

[4 marks]

(b) • Say $\mathfrak{A} = \langle A; R \rangle$. Then $\pi : A \to A$ is an automorphism if π is bijective and for all $a, b \in A$, $R(a,b) \Leftrightarrow R(\pi(a), \pi(b))$

Bookwork

[2 marks]

• We must show that for any formula $\phi(\bar{v})$ of $\mathcal{L}$, and any $\bar{a} \subset A$, $\mathfrak{A} \models \phi[\bar{a}] \Leftrightarrow \mathfrak{A} \models \phi[\pi(\bar{a})]$ .... (*)

The proof of (*) is by induction on ϕ.

Case 1 ϕ atomic.

$\mathfrak{A} \models v_i \doteq v_j [\bar{a}] \Leftrightarrow a_i = a_j$

$\Leftrightarrow \pi(a_i) = \pi(a_j)$ (as π is 1-1),

$\Leftrightarrow \mathfrak{A} \models v_i \doteq v_j [\pi(\bar{a})]$.

$\mathfrak{A} \models P(v_i, v_j)[\bar{a}] \Leftrightarrow R(a_i, a_j)$

$\Leftrightarrow R(\pi(a_i), \pi(a_j))$ (above property of π)

$\Leftrightarrow \mathfrak{A} \models P(v_i, v_j)[\pi(a_i), \pi(a_j)]$.

Inductive step (assume (*) true for χ, ψ).

Case 2 ϕ is $\neg\chi$.

Then $\mathfrak{A} \models \phi[\bar{a}] \Leftrightarrow$ not $\mathfrak{A} \models \chi[\bar{a}]$

$\Leftrightarrow$ not $\mathfrak{A} \models \chi[\pi(\bar{a})]$ (inductive assumption)

$\iff \mathcal{G} \models \phi[\pi(\bar{a})]$.

<u>Case 2</u> ϕ is $(\chi \wedge \psi)$.

Then $\mathcal{G} \models \phi[\bar{a}] \iff \mathcal{G} \models \chi[\bar{a}]$ and $\mathcal{G} \models \psi[\bar{a}]$

$\iff \mathcal{G} \models \chi[\pi(\bar{a})]$ and $\mathcal{G} \models \psi[\pi(\bar{a})]$ (ind. hyp.)

$\iff \mathcal{G} \models \phi[\pi(\bar{a})]$.

<u>Case 3</u> ϕ is $\exists v_i \psi$.

Then $\mathcal{G} \models \exists v_i \psi[\bar{a}] \iff$ for some $b_i \in A$, $\mathcal{G} \models \psi[a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n]$

$\iff$ " " ", $\mathcal{G} \models \psi[\pi(a_1), \ldots, \pi(b_i), \ldots, \pi(a_n)]$

(inductive hypothesis)

But π is surjective, so every element c_i of A has the form $\pi(b_i)$ for some $b_i \in A$. So $\mathcal{G} \models \exists v_i \psi[\bar{a}]$ is equivalent to: for some $c_i \in A$, $\mathcal{G} \models \psi[\pi(a_1), \ldots, \pi(a_{i-1}), c_i, \pi(a_{i+1}), \ldots, \pi(a_n)]$.

$\iff \mathcal{G} \models \phi(\pi(\bar{a}))$.

Bookwork — This completes the proof of (b).

[6 marks]

(c) Let $S \subseteq \mathbb{Z}$ be definable in $\mathcal{G}_0$. So there is some formula $\phi(v_1)$ of $\mathcal{L}$ such that $\forall z \in \mathbb{Z}$, $z \in S \iff \mathcal{G} \models \phi[z]$. Let $a, b \in \mathbb{Z}$. (Say $a \le b$ (w.l.o.g.).) Consider the map $\pi : \mathbb{Z} \to \mathbb{Z}$, $z \mapsto z + (b-a)$. Clearly π is bijective and $\forall z_1, z_2 \in \mathbb{Z}$, $z_1 < z_2 \iff \pi(z_1) < \pi(z_2)$, so π is an automorphism of $\mathcal{G}_0$. Since $\pi(a) = b$, we have by (b) that $\mathcal{G}_0 \models \phi[a] \iff \mathcal{G}_0 \models \phi[\pi(a)] \iff \mathcal{G}_0 \models \phi[b]$.

I.e. $a \in S \iff b \in S$. Since this holds for any $a, b \in \mathbb{Z}$, we must have $S = \emptyset$ or $S = \mathbb{Z}$.

New, but similar examples on problem sheets.

[6 marks]

Ditto. — The same is not true for $\mathcal{G}_1$, since e.g. $S = \{1\}$ is definable, by the formula $\forall v_2 (P(v_1, v_2) \vee v_1 \doteq v_2)$.

[2 marks]

Question 2

- $\mathfrak{A} \preccurlyeq \mathfrak{B} \iff$ for all formulas $\phi(\bar{v})$ of $\mathcal{L}$ and all $\bar{a} \subset \mathrm{dom}(\mathfrak{A})$, we have (dom($\mathfrak{A}$) is a subset of dom($\mathfrak{B}$) and), $\mathfrak{A} \models \phi[\bar{a}] \iff \mathfrak{B} \models \phi[\bar{a}]$.

Bookwork [2 marks]

- Tarski's Lemma: $\mathfrak{A} \preccurlyeq \mathfrak{B} \iff \mathfrak{A} \subseteq \mathfrak{B}$ and whenever $a_1, \ldots, a_{p-1}, a_{p+1}, \ldots, a_n \in A$, $b_p \in B$ and $\phi(v_1, \ldots, v_n)$ is a formula of $\mathcal{L}$ with $\mathfrak{B} \models \phi[a_1, \ldots, a_{p-1}, b_p, a_{p+1}, \ldots, a_n]$, then there is some $a_p \in A$ with $\mathfrak{B} \models \phi[a_1, \ldots, a_{p-1}, a_p, a_{p+1}, \ldots, a_n]$.

Bookwork [3 marks]

- Write $\mathfrak{A} = \langle \mathbb{N}; E \rangle$. Suppose that $\mathfrak{A} \subseteq \mathfrak{B}$. Let $\mathfrak{B} = \langle M; D \rangle$. Then $\mathbb{N} \subseteq M$ and $\mathbb{N} \cap D = E$.

Now let $a_1, \ldots, a_{p-1}, a_{p+1}, \ldots, a_n \in \mathbb{N}$ and $b_p \in M$ be as in ~~the hypotheses of~~ Tarski's Lemma and, likewise, let $\phi(v_1, \ldots, v_n)$ be any formula of $\mathcal{L}$ such that

$$\mathfrak{B} \models \phi[a_1, \ldots, a_{p-1}, b_p, a_{p+1}, \ldots, a_n] \quad \text{---} (*).$$

The idea (suggested by the comment in square parentheses – which is really a hint) is to construct a map $\pi : M \to M$ which is (i) an automorphism of $\mathfrak{B}$; (ii) satisfies $\pi(b_p) \in \mathbb{N}$ and (iii) satisfies $\pi(a_i) = a_i$ for $i = 1, \ldots, p-1, p+1, \ldots, n$.

Then by using part (b) of question 1 we have that

$$\mathfrak{B} \models \phi[a_1, \ldots, a_{p-1}, b_p, a_{p+1}, \ldots, a_n] \iff \mathfrak{B} \models \phi[\pi(a_1), \ldots, \pi(a_{p-1}), \pi(b_p), \pi(a_{p+1}), \ldots, \pi(a_n)]$$
$$\iff \mathfrak{B} \models \phi[a_1, \ldots, a_{p-1}, \pi(b_p), a_{p+1}, \ldots, a_n]$$
(by (iii))

So by (*) we have $\mathcal{B} \models \phi[a_1, \ldots, a_{p-1}, \pi(b_p), a_{p+1}, \ldots, a_n]$, and since $\pi(b_p) \in \mathbb{N}$ (by (iii)), this verifies all the hypotheses of Tarski's condition (i.e. the RHS of the "$\Leftrightarrow$" in Tarski's Lemma). Hence $\mathcal{A} \preccurlyeq \mathcal{B}$ as required.

[7 marks]

Construction of π

If $b_p \in \mathbb{N}$, we just take $\pi = id_M$. (In fact, there is nothing to do.)

If $b_p \notin \mathbb{N}$, there are two cases.

Case 1 $b_p \in D$.

Since E is infinite, we may choose $a_p \in E \setminus \{a_1, \ldots, a_{p-1}, a_{p+1}, \ldots, a_n\}$. We then define, for $m \in M$,

$$\pi(m) = \begin{cases} m & \text{if } m \notin \{b_p, a_p\}. \\ a_p & \text{if } m = b_p \\ b_p & \text{if } m = a_p. \end{cases}$$

Then π is clearly bijective, and $\forall m \in M$, $m \in D \Leftrightarrow \pi(m) \in D$ (note that $D \cap \mathbb{N} = E$). So π is an automorphism which leaves everything except a_p, b_p fixed. In particular $\pi(a_i) = a_i$ for $i = 1, \ldots, p-1, p+1, \ldots, a_n$. Also $\pi(b_p) = a_p \in \mathbb{N}$.

Case 2 $b_p \in M \setminus D$.

Then just choose $a_p \in (\mathbb{N} \setminus E) \setminus \{a_1, \ldots, a_{p-1}, a_{p+1}, \ldots, a_n\}$ and proceed as in case 1.

[8 marks]

A new example but similar examples using this method done in lectures.

Question 3

(a) Let $F_n(T)$ denote the set of all formulas of $\mathcal{L}$ with free variables amongst $v_1, \dots, v_n$.

(i) An <u>n-type</u> (over T) is a subset $p \subseteq F_n(T)$ satisfying
(1) if $\phi, \psi \in p$ then $(\phi \wedge \psi) \in p$;
(2) for all $\phi \in p$, $T \models \exists v_1 \dots \exists v_n \phi$;
(3) for all $\phi \in F_n(T)$, either $\phi \in p$ or $\neg\phi \in p$.

Bookwork [2 marks]

(ii) A n-type (over T) p is called <u>principal</u> if there is some $\phi \in p$ such that for $\psi \in p$, $T \vdash \forall v_1 \dots v_n (\phi \to \psi)$.

Bookwork [1 mark]

(b) The Omitting Types Theorem states that if p is an n-type (over T) which is not principal, then there exists a countable model of T which omits (i.e does not realise) p.

Bookwork [2 marks]

(c) The Ryll-Nardzewski Theorem states that T is $\aleph_0$-categorical (i.e. all models of T of power $\aleph_0$ are isomorphic) if and only if for every n, $F_n(T)$ is finite modulo equivalence in T.

Bookwork [2 marks]

<u>Proof.</u>

$\Leftarrow$: This is the standard back-and-forth argument: let $\mathfrak{A} = \langle A; \dots \rangle$, $\mathfrak{B} = \langle B; \dots \rangle$ be two models of T with card(A) = card$(B) = \aleph_0$. Suppose $\bar{a} = a_1, \dots, a_n \in A$, $\bar{b} = b_1, \dots, b_n \in B$ satisfy (*) $\mathfrak{A} \models \phi[\bar{a}] \iff \mathfrak{B} \models \phi[\bar{b}]$, for all $\phi(\bar{v}) \in F_n(T)$. Let $a_{n+1} \in A$. One finds $b_{n+1} \in B$ to satisfy (*) for $\bar{a}^\frown a_{n+1}$, $\bar{b}^\frown b_{n+1}$ as follows. Let $\chi(\bar{v}, v_{n+1})$ be a <u>principal formula</u> for $\bar{a}^\frown a_{n+1}$ (i.e. a generator for the type of $\bar{a}^\frown a_{n+1}$, which clearly exists since $F_{n+1}(T)$ is finite). Then $\mathfrak{A} \models \exists v_{n+1} \chi[\bar{a}]$, so $\mathfrak{B} \models \exists v_{n+1} \chi[\bar{b}]$ by (*).

Then any b_{n+1} witnessing v_{n+1} in $\mathcal{B}$ will work.

Now enumerate A, B in ω-sequences and do the standard inductive construction of an isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

[5 marks]

This proof is bookwork, but considerable judgement is required in setting down the crucial points.

$\Rightarrow$: Suppose $F_n(T)$ is not finite (modulo T) for some n. Then one shows that the set $S := \{\neg\phi(\bar{v}) : \phi(\bar{v}) \in F_n(T)$ a principal formula of $T\}$ has the "finite intersection property" (modulo T). Hence S extends to an n-type p (over T), which is necessarily non-principal. Hence, by the Omitting Types Theorem, there is a countable model $\mathcal{C} \models T$ omitting p. But for any type (over T), there is always a countable model realising it. So if $\mathcal{B} \models T$, $\mathcal{B}$ countable, realises p we cannot have $\mathcal{C} \cong \mathcal{B}$. So T is not $\aleph_0$-categorical.

[5 marks]

(d) For each $m \geq 1$, let $\phi_m(v_1, v_2)$ be the formula (in the language for groups) $v_1^m \simeq v_2$ (where v_1^m denotes the term $\underbrace{v_1 \circ (v_1 \circ \cdots \circ v_1)}_{m\text{-times}}$).

New, but similar example done in lectures

If $T = Th(\mathcal{G})$ is $\aleph_0$-categorical, then $F_2(T)$ is finite (modulo T), so there exist m, l, $m \neq l$, such that $T \models \forall v_1 \forall v_2 (\phi_m(v_1, v_2) \leftrightarrow \phi_l(v_1, v_2))$. This clearly implies $\mathcal{G} \models \forall v_1\, v_1^m = v_1^l$, so for all $g \in \text{dom}(\mathcal{G})$ $g^m = g^l$, i.e. $g^{|m-l|} = e$. So take $N = |m-l|\ (\neq 0)$.

[3 marks]

Question 4

Firstly, the quantifier-free formulas of $\mathcal{L}$ are defined inductively:

- every atomic formula is quantifier-free;
- if ϕ, ψ are quantifier-free, then so are $\neg\phi$ and $(\phi \wedge \psi)$.

Bookwork

[2 marks]

A universal formula is now defined to be one of the form $\forall v_{i_1} \ldots \forall v_{i_r} \phi$, where ϕ is quantifier-free.

Bookwork

[1 mark]

(a) We prove the assertion by induction on the number (r in the above) of universal quantifiers. We are given the result for $r = 0$.

Now, $\mathfrak{B} \models \forall v_{i_1} \ldots \forall v_{i_r} \phi\, [a_1, \ldots, a_n]$

$\iff$ for all $b_{i_1} \in B$, $\mathfrak{B} \models \forall v_{i_2} \ldots \forall v_{i_r} \phi [a_1, \ldots, a_n](a_{i_1}/b_{i_1})$

$\implies$ for all $b_{i_1} \in A$, $\mathfrak{B} \models$ —— " ——
(since $A \subseteq B$)

$\implies$ for all $b_{i_1} \in A$, $\mathfrak{A} \models$ —— " ——
(by inductive hypothesis)

$\iff$ $\mathfrak{A} \models \forall v_{i_1} \forall v_{i_2} \ldots \forall v_{i_r} \phi\, [a_1, \ldots, a_n]$,
as required.

Bookwork

[3 marks]

(b) Let $A_0 := \{ \tau^{\mathfrak{A}}[a] : \tau(v_i) \text{ a term of } \mathcal{L} \}$.

We only need to show that A_0 is closed under the basic functions of $\mathfrak{A}$, and that it contains the distinguished elements of $\mathfrak{A}$. For then A_0 is the domain of a (unique) substructure of $\mathfrak{A}$, call it $\mathfrak{A}_0$, and $\mathfrak{A}_0$, by definition, has the required property.

New in this context, but the term model has been studied at length.

Now every distinguished element d (of $\mathfrak{A}$) certainly has the form $c^{\mathfrak{A}}$ for some constant symbol c of $\mathcal{L}$.

Further, if f is a p-ary function of the signature of $\mathfrak{A}$, and F is the corresponding p-ary function symbol of $\mathcal{L}$, and $\tau_1^{\mathfrak{A}}[a], \ldots, \tau_p^{\mathfrak{A}}[a] \in A_0$, then $f(\tau_1^{\mathfrak{A}}[a], \ldots, \tau_p^{\mathfrak{A}}[a]) = F(\tau_1, \ldots, \tau_p)^{\mathfrak{A}}[a]$ (by definition of interpretation of terms), and the RHS here is visibly in A_0. Hence A_0 is closed under f, as required.

[6 marks]

(c) Let γ be a new constant symbol (not in $\mathcal{L}$).

Suppose, for a contradiction, that

$$T \cup \{\neg\phi(\gamma, \tau(\gamma)) : \tau(v_1) \text{ a term of } \mathcal{L}\} := T'$$

is finitely satisfiable.

Then by the Compactness Theorem it has a model, $\mathfrak{A}'$ say. ($\mathfrak{A}'$ is a $\mathcal{L} \cup \{\gamma\}$-structure). Let $a = \gamma^{\mathfrak{A}'}$ and let $\mathfrak{A}_0$ be as in (b), (as an $\mathcal{L}$-substructure of $\mathfrak{A}' \upharpoonright \mathcal{L}$.)

Since ϕ is quantifier-free we have (by (a)), that $\mathfrak{A}_0 \models \neg\phi[a, \tau^{\mathfrak{A}}[a]]$ for each term $\tau(v_1)$ of $\mathcal{L}$.

This is new, but similar (and deeper) arguments have been done in the context of preservation theorems.

But every $x \in \mathrm{dom}(\mathfrak{A}_0)$, has the form $\tau^{\mathfrak{A}}[a]$, so $\mathfrak{A}_0 \models \forall v_2 \neg\phi(v_1, v_2)[a]$, and hence $\mathfrak{A}_0 \models \exists v_1 \forall v_2 \neg\phi(v_1, v_2)$, or $\mathfrak{A}_0 \models \neg\forall v_1 \exists v_2 \phi(v_1, v_2)$. But as T is universal and $\mathfrak{A}_0 \subseteq \mathfrak{A}' \upharpoonright \mathcal{L}$, and $\mathfrak{A}' \upharpoonright \mathcal{L} \models T$, we have (by (a)) that $\mathfrak{A}_0 \models T$, contradicting the fact that $T \models \forall v_1 \exists v_2 \phi(v_1, v_2)$.

So some finite subset of T' is not satisfiable. So there exists N s.t. $T \models \neg\left(\bigwedge_{1 \le i \le N} \neg\phi(\gamma, \tau_i(\gamma))\right)$, for some terms $\tau_1(v_1), \ldots, \tau_N(v_1)$ of $\mathcal{L}$. Now since γ is a constant symbol not in $\mathcal{L}$, we may universally quantify over it and then boolean rearrangement gives the result.

[8 marks]