MODEL THEORY       MATH 43052/63052

Solution exam mark scheme

Question 1

(a) (i) The only terms are variables.
(ii) So the only atomic formulas are of the form \( v_i = v_j \) or \( P(v_i, v_j) \).
(iii) Every atomic formula is a formula:
     \[ \neg \phi, \psi \text{ are formulas, so are } \neg \phi, (\phi \land \psi), \exists v_i \phi. \]

(b) Say \( G = \langle A, R \rangle \). Then \( \pi : A \to A \) is an automorphism if \( \pi \) is bijective and for all \( a, b \in A \), \( R(a, b) \iff R(\pi(a), \pi(b)) \).

We must show that for any formula \( \phi(\bar{x}) \) of \( L \), and any \( \bar{a} \in A \), \( G \models \phi[\bar{a}] \iff G \models \phi[\pi(\bar{a})] \) (x).

The proof of (x) is by induction on \( \phi \).

Case 1: \( \phi \) atomic.
\[ G \models v_i = v_j[\bar{a}] \iff a_i = a_j. \]
\[ \iff \pi(a_i) = \pi(a_j) \quad (\text{as } \pi \text{ is } 1-1), \]
\[ \iff G \models v_i = v_j[\pi(\bar{a})]. \]

\[ G \models P(v_i, v_j)[\bar{a}] \iff R(a_i, a_j). \]
\[ \iff R(\pi(a_i), \pi(a_j)) \quad (\text{since property of } \pi), \]
\[ \iff G \models P(v_i, v_j)[\pi(\bar{a})]. \]

Inductive step (assume (x) true for \( \chi, \psi \)).

Case 2: \( \phi \) is \( \neg \chi \).

Then \( G \models \phi[\bar{a}] \iff \neg G \models \chi[\bar{a}] \iff \not \models G \models \chi[\pi(\bar{a})] \quad (\text{inductive assumption}). \]
\[
\Leftrightarrow \quad \mathcal{G} \models \phi [\pi(\bar{a})].
\]

**Case 2.** \( \phi \) in \( \langle \mathcal{K}, \land, \forall \rangle \).

Then
\[
\mathcal{G} \models \phi [\bar{a}] \Leftrightarrow \mathcal{G} \models \mathcal{K}[\bar{a}] \quad \text{and} \quad \mathcal{G} \models \mathcal{A}[\bar{a}]
\]
\[
\Leftrightarrow \quad \mathcal{G} \models \mathcal{K}[\pi(\bar{a})] \quad \text{and} \quad \mathcal{G} \models \mathcal{A}[\pi(\bar{a})] \quad \text{(ind. hyp.)}
\]
\[
\Leftrightarrow \quad \mathcal{G} \models \phi [\pi(\bar{a})].
\]

**Case 3.** \( \phi \) is \( \exists \mathcal{K} \mathcal{A} \).

Then
\[
\mathcal{G} \models \exists \mathcal{K} \mathcal{A}[\bar{a}] \Leftrightarrow \exists \text{ some } a_2 \in A, \mathcal{G} \models \mathcal{K}[\mathcal{A}[a_1, a_2, \ldots, a_m]] \quad \text{for some } a_1 \in A.
\]
\[
\Leftrightarrow \quad \mathcal{G} \models \phi [\pi(\bar{a})] \quad \text{(inductive hypothesis)}
\]

But \( \pi \) is surjective, so every element \( a_2 \) of \( A \) has the form \( \pi(b_2) \) for some \( b_2 \in B \). So \( \mathcal{G} \models \exists \mathcal{K} \mathcal{A}[\bar{a}] \) is equivalent to
\[
\mathcal{G} \models \phi [\pi(\bar{a})].
\]


This completes the proof of (b).

(c) \( \forall S \subseteq \mathbb{Z} \) is definable in \( \mathbb{G}_0 \). So there is some formula \( \phi(\mathcal{V}) \) of \( \mathbb{L} \) such that \( \forall z \in \mathbb{Z}, z \in S \iff \mathcal{G} \models \phi(z) \).

Let \( a, b \in \mathbb{Z} \). (Say \( a \leq b \) (w.l.o.g.).) Consider the map \( \pi : \mathbb{Z} \to \mathbb{Z}, z \mapsto z + (b-a) \). Clearly \( \pi \) is bijective and \( \forall z_1, z_2 \in \mathbb{Z}, z_1 \leq z_2 \iff \pi(z_1) \leq \pi(z_2) \). So \( \pi \) is an automorphism of \( \mathbb{G}_0 \). Since \( \pi(a) = b \), we have by (b)
\[
\text{that } \mathcal{G}_0 \models \phi[a] \quad \Rightarrow \quad \mathcal{G}_0 \models \phi[\pi(a)] \quad \Rightarrow \quad \mathcal{G}_0 \models \phi[b],
\]

i.e., \( a \in S \iff b \in S \). Since this holds for any \( a, b \in \mathbb{Z} \), we must have \( S = \emptyset \) or \( S = \mathbb{Z} \).

\[ \text{[6 marks]} \]

**Note.** The same is not true for \( \mathbb{G}_1 \), since e.g., \( S = \{3\} \) is definable, by the formula \( \forall z_2 (\phi(v_1, v_2) \vee v_1 = v_2) \).

\[ \text{[2 marks]} \]
Question 2

* \( G \preceq L \iff \) for all formulas \( \phi(v) \) of \( L \) and all \( \overline{a} \in \text{dom}(G) \), we have (\( \text{dom}(G) \) is a subset of \( \text{dom}(L) \) and) \( \models_G \phi[\overline{a}] \iff L \models \phi[\overline{a}] \). [2 marks]

* Tanki's Lemma: \( G \preceq L \iff G \preceq L \) and whenever \( a_1, \ldots, a_{p-1}, a_p, a_{p+1}, \ldots, a_n \in A \), \( b_p \in B \) and \( \phi(v_1, \ldots, v_m) \) is a formula of \( L \), with \( L \models \phi[a_1, \ldots, a_{p-1}, b_p, a_{p+1}, \ldots, a_n] \), then there is some \( a_p \in A \) with \( L \models \phi[a_1, \ldots, a_{p-1}, a_p, a_{p+1}, \ldots, a_n] \). [2 marks]

* Write \( G = \langle M, E \rangle \), Suppose that \( G \preceq L \).

Let \( L = \langle M, D \rangle \). Then \( M \subseteq M \) and \( MN \cup D = E \).

Now let \( a_1, \ldots, a_{p-1}, a_p, a_{p+1}, \ldots, a_n \in IN \) and \( b_p \in M \) be as in the hypothesis of Tanki's lemma and, likewise, let \( \phi(v_1, \ldots, v_m) \) be any formula of \( L \) such that

\[ L \models \phi[a_1, \ldots, a_{p-1}, b_p, a_{p+1}, \ldots, a_n] \quad \cdots \quad (*) \]

The idea (suggested by the comment in square parentheses - which is really a hint) is to construct a map \( \pi : M \rightarrow M \) which is (i) an automorphism of \( G \); (ii) satisfies \( \pi(b_p) \in \text{IN} \) and (iii) satisfies \( \pi(a_i) = a_i \) for \( i = 1, \ldots, p, p+1, \ldots, n \).

Then by using part (b) of question 1 we have that

\[ L \models \phi[a_1, \ldots, a_{p-1}, b_p, a_{p+1}, \ldots, a_n] \iff L \models \phi[\pi(a_1), \ldots, \pi(a_{p-1}), \pi(b_p), \pi(a_{p+1}), \ldots, \pi(a_n)] \iff L \models \phi[a_1, \ldots, a_{p-1}, \pi(b_p), a_{p+1}, \ldots, a_n] \quad (\text{by (iii)}) \]
See by (c) we have \( \mathfrak{D} = \phi[a_1, a_2, \ldots, a_n] \),
and since \( \Pi(b_p) \in \mathbb{N} \) (by (iii)), this verifies all
the hypotheses of Tarski's conditions (i.e. the RHS of
the \( \Rightarrow \) in Tarski's Lemma). Hence \( a \leq \mathfrak{D} \) as required.

[7 marks]

Construction of \( \Pi \)

- If \( b_p \in \mathbb{N} \), we just take \( \Pi = \text{id}_E \). (In fact, there is
  nothing to do.)
- If \( b_p \notin \mathbb{N} \), there are two cases.

Case 1: \( b_p \in D \).

Since \( E \) is infinite, we may choose \( a_p \in E \setminus \mathbb{Z} \).

Then define, for \( m \in M \),

\[
\Pi(m) = \begin{cases} 
  m, & \text{if } m \notin \{b_p, a_p^2\} \\
  a_p, & \text{if } m = b_p \\
  b_p, & \text{if } m = a_p.
\end{cases}
\]

Then \( \Pi \) is clearly bijective, and \( \forall m \in M, m \in D \Rightarrow \Pi(m) \in D \)
(note that \( 0 \in \mathbb{N} \subseteq E \)). So \( \Pi \) is an automorphism
which leaves everything except \( a_p, b_p \) fixed. In particular,
\( \Pi(a_i) = a_i \) for \( i = 1, \ldots, p-1, p+1, \ldots, a_n \). Also \( \Pi(b_p) = a_p \in \mathbb{N} \).

Case 2: \( b_p \notin M \setminus D \).

Then just choose \( a_p \in (E \setminus \mathbb{Z}) \setminus \{a_1, a_2, \ldots, a_n\} \) and
proceed as in Case 1.

[8 marks]
Question 3

(a) Let \( F_n(T) \) denote the set of all formulas of \( T \) with free variables amongst \( x_1, \ldots, x_n \).

(i) An \( n \)-type (over \( T \)) \( p \) is a subset \( \varphi \in F_n(T) \) satisfying:

\begin{enumerate}
  \item if \( \varphi, \psi \in \varphi \text{ then } (\varphi \lor \psi) \in \varphi \); \[ 2 \text{ marks} \]
  \item for all \( \varphi \in \varphi \), \( T \models \exists x_1 \ldots \exists x_n \varphi \); \[ 1 \text{ mark} \]
  \item for all \( \varphi \in F_n(T) \), either \( \varphi \in \varphi \) or \( \neg \varphi \in \varphi \).
\end{enumerate}

(ii) A \( n \)-type (over \( T \)) \( p \) is called principal if there is some \( \psi \in \varphi \) such that \( \forall \psi \in \varphi \), \( T \models \forall x_1 \ldots \forall x_n (\psi \rightarrow \psi) \).

(iii) The Omitting Types Theorem states that if \( p \) is an \( n \)-type (over \( T \)) which is not principal, then there exists a countable model of \( T \) which omits \( p \) (i.e., does not realise) \( p \).

(c) The Ryll-Nardzewski Theorem states that \( T \) is \( \aleph_0 \)-categorical (i.e., all models of \( T \) of power \( \aleph_0 \) are isomorphic) if and only if for every \( n \), \( F_n(T) \) is finite modulo equivalence in \( T \).

Proof:

\( \leftarrow \): This is the standard back-and-forth argument: let \( \mathfrak{A} = \langle A_1, \ldots \rangle \), \( \mathfrak{B} = \langle B_1, \ldots \rangle \) be two models of \( T \) with \( \text{card}(\mathfrak{A}) = \text{card}(\mathfrak{B}) = \aleph_0 \). Suppose \( \hat{a} = \bar{a}_1, \ldots, \bar{a}_n \in A_1, \ldots, A_n \in \mathfrak{B} \) satisfy \( (\forall \varphi \in F_1(T)) \mathfrak{A} \models \varphi[\bar{a}] \). We get \( \mathfrak{B} \models \varphi[\bar{a}] \) for all \( \varphi \in F(T) \).

Let \( \bar{a}_1, \ldots, \bar{a}_n \) be a finite subset of \( A_1, \ldots, A_n \). One finds \( \bar{b}_1, \ldots, \bar{b}_n \in B_1, \ldots, B_n \) to satisfy \( (\forall \varphi \in F_n(T)) \mathfrak{B} \models \varphi[\bar{b}] \) for \( \bar{b}_1, \ldots, \bar{b}_n \). 

The back-and-forth condition is satisfied if \( \varphi \in F_1(T) \) is a principal formula for \( \bar{a}_1, \ldots, \bar{a}_n \) (i.e., a generator for the type of \( \bar{a}_1, \ldots, \bar{a}_n \), which clearly exists since \( F_n(T) \) is finite). If \( \mathfrak{A} \models \varphi[\bar{a}] \), then \( \mathfrak{B} \models \varphi[\bar{b}] \) for \( \bar{b} \in F_1(T) \) by \((\ast)\).
Then any \( h_{n+1} \) witnessing \( v_{m+1} \) in \( \phi_{n+1} \) will work.

Now assume \( A, B \) in \( \omega \)-sequences and do the standard inductive construction of an isomorphism from \( A \) to \( B \).

[5 marks]

\[ \Rightarrow : \text{ Suppose } F_n(T) \text{ is not finite (modulo } T \text{) for some } n. \text{ Then one shows that the set } \]

\[ S = \{ \phi(v) : \phi(v) \in F_n(T) \text{ a principal formula of } T \} \]

has the "finite intersection property" (modulo \( T \)). Hence \( S \) extends to an type \( p \) (over \( T \)) which is necessarily non-principal. Hence, by the Omitting

Type Theorem, there is a countable model \( A \models T \)

omitting \( p \). But for any type (over \( T \)), there is always a countable model realizing it. So if \( \phi(T) \), \( \text{Countable, realizing } p \), we cannot have \( A \models \phi(T) \).

So \( T \) is not \( \omega \)-categorical.

[5 marks]

(1) \( \text{ For each } m \geq 1, \text{ let } \phi_m(v_1, v_2) \text{ be the formula (in the language for groups) } v_1^m = v_2 \text{ (where } v_1^m \text{ denotes the } m \text{-th power).} \]

If \( T = \text{Th}(G) \text{ is } \omega \)-categorical, then \( F_2(T) \) is finite (modulo \( T \)), so there exist \( m, l, m + l \), such that \( T \models \forall v_1 \forall v_2 (\phi_m(v_1, v_2) \iff \phi_l(v_1, v_2)) \). This clearly implies \( g \models \forall v_1 v_2^m = v_2 \), so for all \( g \in \text{dom} (G) \)

\[ g^m = g^l, \text{ i.e. } g^{m-l} = 1 \text{. So take } N = m - l \text{ (not 0).} \]

[3 marks]
Question 4

Finally, the quantifier-free formulas of $L$ are defined inductively:
- Every atomic formula is quantifier-free;
- If $\phi, \psi$ are quantifier-free, then so are $\neg \phi$ and $(\phi \lor \psi)$.

A universal formula is now defined to be one of the form $\forall x_1 \ldots \forall x_n \phi$, where $\phi$ is quantifier-free.

(a) We prove the assertion by induction on the number (i in the above) of universal quantifiers. We are given the result for $i = 0$.

Now, $\models \forall x_1 \ldots \forall x_n \phi \ [a_1, \ldots, a_n]$

$\iff$ for all $b_1 \in B$, $\models \forall x_1 \ldots \forall x_n \phi[x_1 \mapsto a_1, \ldots, x_n \mapsto a_n, b_1 / x_i]$

$\iff$ for all $b_1 \in B$, $\models \forall x_i, \forall x_1, \ldots, \forall x_{i-1}, \forall x_{i+1} \ldots \forall x_n \phi[\alpha_i, a_i, \beta_i][x_i / x_i]$

$\iff$ for all $b_i \in A$, $\models \alpha_i$ (and $A \subseteq B$)

$\iff$ for all $b_i \in A$, $\models \forall x_i, \forall x_1, \ldots, \forall x_{i-1}, \forall x_{i+1} \ldots \forall x_n \phi[\alpha_i, a_i, \beta_i][x_i / x_i]$

$\iff$ as required.

(b) Let $A_0 := \Sigma \in \omega^e : \exists (v) a term of x_2^3$.

We only need to show that $A_0$ is closed under the basic functions of $\omega^e$, and that it contains the distinguished elements of $\omega^e$. For then $A_0$ is the domain of an (unique) substructure of $\omega^e$, call it $\omega_0$, and $\omega_0$, by definition, has the required property.
Now, given the diagram, for some constant $\alpha$.

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