Two and a half hours

THE UNIVERSITY OF MANCHESTER

MODEL THEORY

Date

Time

Answer all the questions. Each question is worth 20 marks. A further 20 marks are available from work during the semester, making a total of 100.

Electronic calculators may be used, provided that they cannot store text.

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1. Let $\mathcal{L}$ be the language containing just one binary relation symbol P.

(i) Give precise definitions, **in the case of this particular language**, for (a) the set of *terms* of $\mathcal{L}$, (b) the set of *atomic formulas* of $\mathcal{L}$, and (c) the set of *formulas* of $\mathcal{L}$.

(ii) Let $\mathfrak{A}$ be an $\mathcal{L}$-structure with $dom(\mathfrak{A})=A$. What does it mean to say that a map $\pi : A \to A$ is an *automorphism* of $\mathfrak{A}$? Prove that an automorphism preserves the truth of all formulas.

(iii) Now let $\mathfrak{A}_0$, $\mathfrak{A}_1$ be the $\mathcal{L}$-structures $\langle \mathbb{Z}; < \rangle$, $\langle \mathbb{N}; < \rangle$ respectively, where $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and the set of positive integers respectively, and $<$ denotes the usual ordering. Prove that a subset S of $\mathbb{Z}$ is definable in $\mathfrak{A}_0$ if and only if $S=\emptyset$ or $S=\mathbb{Z}$. Is the same true for the structure $\mathfrak{A}_1$? Give reasons for your answer.

[20 marks]

2. Let $\mathcal{L}$ be a language and let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures. Assume that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$. What does it mean to say that $\mathfrak{A}$ is an *elementary substructure* of $\mathfrak{B}$ (i.e. $\mathfrak{A} \preceq \mathfrak{B}$)? State *Tarski's Lemma* giving necessary and sufficient conditions for $\mathfrak{A}$ to be an elementary substructure of $\mathfrak{B}$.

Suppose now that $\mathcal{L}$ consists of just one unary relation symbol P, and let $\mathfrak{A}$ be the $\mathcal{L}$-structure with domain the set of positive integers and with P interpreted as the set of even positive integers. Prove that if $\mathfrak{B}$ is any $\mathcal{L}$-structure with $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \preceq \mathfrak{B}$. [You may assume the result of part (ii) of question 1.]

[20 marks]

3. Let $\mathcal{L}$ be a countable language and T a complete $\mathcal{L}$-theory without finite models.

(a) Explain what is meant by (i) an *n-type* (over T), and (ii) a *principal* n-type (over T).

(b) State the Omitting Types Theorem.

(c) State Ryll-Nardzewski's Theorem giving a necessary and sufficient condition for T to be $\aleph_0$-categorical, and briefly describe the main ideas involved in its proof.

(d) Let $\mathcal{G}$ be an infinite group whose complete theory is $\aleph_0$-categorical. Prove that there exists a positive integer N such that for all $g \in dom(\mathcal{G})$ we have that $g^N = e$, where e denotes the identity element of $\mathcal{G}$.

[20 marks]

4. Let $\mathcal{L}$ be a language. What is meant by a *universal* formula of $\mathcal{L}$. [You may take as given the notion of a quantifier-free formula of $\mathcal{L}$]

(a) Prove that if $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{L}$-structures with $\mathfrak{A} \subseteq \mathfrak{B}$, then for all universal formulas $\phi(v_1, \ldots, v_n)$ of $\mathcal{L}$ and all $a_1, \ldots, a_n \in dom(\mathfrak{A})$, if $\mathfrak{B} \models \phi[a_1, \ldots, a_n]$ then $\mathfrak{A} \models \phi[a_1, \ldots, a_n]$. [You may assume that this is true in the case that ϕ is quantifier-free.]

(b) Let $\mathfrak{A}$ be any $\mathcal{L}$-structure and suppose that $a \in \mathrm{dom}(\mathfrak{A})$. Prove that there exists a substructure $\mathfrak{A}_\circ$ of $\mathfrak{A}$ such that for every $x \in dom(\mathfrak{A}_\circ)$, there exists a term $\tau(v_1)$ of $\mathcal{L}$ such that $x = \tau^{\mathfrak{A}}[a]$.

(c) Deduce that if T is a set of universal sentences of $\mathcal{L}$ and $\phi(v_1, v_2)$ is a quantifier-free formula of $\mathcal{L}$ such that $T \models \forall v_1 \exists v_2 \phi(v_1, v_2)$, then there exist $N \geq 1$, and terms $\tau_1(v_1), \ldots, \tau_N(v_1)$ of $\mathcal{L}$ such that $T \models \forall v_1(\phi(v_1, \tau_1(v_1)) \vee \phi(v_1, \tau_2(v_1)) \vee \ldots \vee \phi(v_1, \tau_1(v_N)))$. [Hint: Assume false and use the Compactness Theorem.]

[20 marks]