Two and a half hours

THE UNIVERSITY OF MANCHESTER

MODEL THEORY

24 January 2014

09.45 - 12.15

Answer FOUR of the FIVE questions. If more than FOUR questions are attempted, then credit will be given for the best FOUR answers

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Electronic calculators may be used, provided that they cannot store text.

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1. Let $\mathcal{L}$ be the language containing just one unary function symbol F.

(a) Give precise definitions, in the case of this particular language, for (i) the set of *terms* of $\mathcal{L}$, (ii) the set of *atomic formulas* of $\mathcal{L}$, and (iii) the set of *formulas* of $\mathcal{L}$.

(b) Let $\mathfrak{A}$ be an $\mathcal{L}$-structure with $dom(\mathfrak{A})=A$. What does it mean to say that a function $\pi : A \to A$ is an *automorphism* of $\mathfrak{A}$? Prove that if $\pi : A \to A$ is an automorphism of $\mathfrak{A}$ then all formulas are π-preserved from $\mathfrak{A}$ to $\mathfrak{A}$.

(c) Now let $\mathfrak{A} = \langle \mathbb{Z}; s\rangle$, where $s(n) = n + 1$ for $n \in \mathbb{Z}$. Prove that the only $\mathcal{L}$-definable subsets of $\mathbb{Z}$ are the empty set and $\mathbb{Z}$ itself.

[20 marks]

2. Let $\mathcal{L}$ be a language and let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures. What does it mean to say that $\mathfrak{A}$ is an *elementary substructure* of $\mathfrak{B}$ (i.e. $\mathfrak{A} \preceq \mathfrak{B}$)? State *Tarski's Lemma* giving necessary and sufficient conditions for $\mathfrak{A}$ to be an elementary substructure of $\mathfrak{B}$.

Suppose now that $\mathcal{L}$ contains just a binary relation symbol and consider the $\mathcal{L}$-structure $\mathfrak{R} := \langle \mathbb{R}; <\rangle$.

Suppose that $r_1 < r_2 < \cdots < r_n$ and $s_1 < s_2 < \cdots < s_n$ are two increasing sequences of real numbers. Write down a formula for an automorphism π: $\mathfrak{R} \cong \mathfrak{R}$ such that $\pi(r_i) = \pi(s_i)$ for each $i = 1, \ldots, n$. Deduce that if $\mathfrak{A}$ is any dense linear ordering without endpoints such that $\mathfrak{A} \subseteq \mathfrak{R}$, then $\mathfrak{A} \preceq \mathfrak{R}$. [You may assume that all formulas are preserved by automorphisms.]

[20 marks]

3. Let $\mathcal{L}$ be a countable language and T a complete $\mathcal{L}$-theory without finite models.

(a) What is meant by (i) an n-type (over T), and (ii) a principal n-type (over T)?

(b) Prove that a principal n-type (over T) is realised in all models of T and that any n-type (over T) is realised in some countable model of T.

(c) State the Omitting Types Theorem.

(d) Suppose that there exists a nonprincipal 1-type (over T). Prove that T is not ω-categorical.

(e) Deduce that the complete theory of the structure $\langle \mathbb{N}; s\rangle$, where $s(n) = n + 1$ for $n \in \mathbb{N}$, is not ω-categorical.

[20 marks]

4. Let $\mathcal{L}$ be the language containing just one unary relation symbol P. The aim of this question is to establish the 0-1 law for sentences of $\mathcal{L}$.

(a) For N a positive integer, how many $\mathcal{L}$-structures are there with domain $\{1, \ldots, N\}$?

For τ any sentence of $\mathcal{L}$, let $p(\tau, N)$ denote the proportion of these structures that satisfy τ.

(b) For $n \geq 2$, let ϕ_n be the sentence

$$\exists v_1, \ldots, v_n(\bigwedge_{1 \leq i \neq j \leq n} (\neg v_i \doteq v_j \wedge P(v_i)))$$

and let ψ_n be the sentence

$$\exists v_1, \ldots, v_n(\bigwedge_{1 \leq i \neq j \leq n} (\neg v_i \doteq v_j \wedge \neg P(v_i))).$$

Prove that, for each fixed n,

$$p((\phi_n \wedge \psi_n), N) \to 1$$

as $N \to \infty$.

(c) By using Vaught's test, or otherwise, prove that the set $\{(\phi_n \wedge \psi_n) : n \geq 2\}$ of $\mathcal{L}$-sentences is complete.

(d) Deduce that for any $\mathcal{L}$-sentence τ, either $p(\tau, N) \to 1$ as $N \to \infty$, or $p(\tau, N) \to 0$ as $N \to \infty$.

[20 marks]

5. Explain (without proofs) the method of diagrams and the method of complete diagrams.

Let $\mathcal{L}$ be a language and let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures with $\mathfrak{A} \subseteq \mathfrak{B}$. Assume that for all existential formulas $\phi(v_1, \ldots, v_n)$ of $\mathcal{L}$ and all $a_1, \ldots, a_n \in \mathrm{dom}(\mathfrak{A})$,

$$\mathfrak{A} \models \phi[a_1, \ldots, a_n] \Longleftrightarrow \mathfrak{B} \models \phi[a_1, \ldots, a_n].$$

Prove that there exists an $\mathcal{L}$-structure $\mathfrak{C}$ such that $\mathfrak{B} \subseteq \mathfrak{C}$ and $\mathfrak{A} \preceq \mathfrak{C}$.

[20 marks]

END OF EXAMINATION PAPER