Answer FOUR of the FIVE questions. If more than FOUR questions are attempted, then credit will be given for the best FOUR answers.

Electronic calculators may be used, provided that they cannot store text.
1. Let $\mathcal{L}$ be the language containing just one unary function symbol $F$.

(a) Give precise definitions, in the case of this particular language, for (i) the set of terms of $\mathcal{L}$, (ii) the set of atomic formulas of $\mathcal{L}$, and (iii) the set of formulas of $\mathcal{L}$.

(b) Let $\mathfrak{A}$ be an $\mathcal{L}$-structure with $\text{dom} (\mathfrak{A}) = A$. What does it mean to say that a function $\pi : A \to A$ is an automorphism of $\mathfrak{A}$? Prove that if $\pi : A \to A$ is an automorphism of $\mathfrak{A}$ then all formulas are $\pi$-preserved from $\mathfrak{A}$ to $\mathfrak{A}$.

(c) Now let $\mathfrak{A} = \langle \mathbb{Z}; s \rangle$, where $s(n) = n + 1$ for $n \in \mathbb{Z}$. Prove that the only $\mathcal{L}$-definable subsets of $\mathbb{Z}$ are the empty set and $\mathbb{Z}$ itself.

[20 marks]

2. Let $\mathcal{L}$ be a language and let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures. What does it mean to say that $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$ (i.e. $\mathfrak{A} \preceq \mathfrak{B}$)? State Tarski’s Lemma giving necessary and sufficient conditions for $\mathfrak{A}$ to be an elementary substructure of $\mathfrak{B}$.

Suppose now that $\mathcal{L}$ contains just a binary relation symbol and consider the $\mathcal{L}$-structure $\mathfrak{R} := \langle \mathbb{R}; < \rangle$.

Suppose that $r_1 < r_2 < \cdots < r_n$ and $s_1 < s_2 < \cdots < s_n$ are two increasing sequences of real numbers. Write down a formula for an automorphism $\pi : \mathfrak{R} \cong \mathfrak{R}$ such that $\pi(r_i) = \pi(s_i)$ for each $i = 1, \ldots, n$. Deduce that if $\mathfrak{A}$ is any dense linear ordering without endpoints such that $\mathfrak{A} \subseteq \mathfrak{R}$, then $\mathfrak{A} \preceq \mathfrak{R}$. [You may assume that all formulas are preserved by automorphisms.]

[20 marks]

3. Let $\mathcal{L}$ be a countable language and $T$ a complete $\mathcal{L}$-theory without finite models.

(a) What is meant by (i) an $n$-type (over $T$), and (ii) a principal $n$-type (over $T$)?

(b) Prove that a principal $n$-type (over $T$) is realised in all models of $T$ and that any $n$-type (over $T$) is realised in some countable model of $T$.

(c) State the Omitting Types Theorem.

(d) Suppose that there exists a nonprincipal 1-type (over $T$). Prove that $T$ is not $\omega$-categorical.

(e) Deduce that the complete theory of the structure $\langle \mathbb{N}; s \rangle$, where $s(n) = n + 1$ for $n \in \mathbb{N}$, is not $\omega$-categorical.

[20 marks]
4. Let $\mathcal{L}$ be the language containing just one unary relation symbol $P$. The aim of this question is to establish the 0-1 law for sentences of $\mathcal{L}$.

(a) For $N$ a positive integer, how many $\mathcal{L}$-structures are there with domain $\{1, \ldots, N\}$?

For $\tau$ any sentence of $\mathcal{L}$, let $p(\tau, N)$ denote the proportion of these structures that satisfy $\tau$.

(b) For $n \geq 2$, let $\phi_n$ be the sentence

$$\exists v_1, \ldots, v_n \left( \bigwedge_{1 \leq i \neq j \leq n} (\neg v_i \subseteq v_j \land P(v_i)) \right)$$

and let $\psi_n$ be the sentence

$$\exists v_1, \ldots, v_n \left( \bigwedge_{1 \leq i \neq j \leq n} (\neg v_i \subseteq v_j \land \neg P(v_i)) \right).$$

Prove that, for each fixed $n$,

$$p((\phi_n \land \psi_n), N) \to 1$$

as $N \to \infty$.

(c) By using Vaught’s test, or otherwise, prove that the set $\{ (\phi_n \land \psi_n) : n \geq 2 \}$ of $\mathcal{L}$-sentences is complete.

(d) Deduce that for any $\mathcal{L}$-sentence $\tau$, either $p(\tau, N) \to 1$ as $N \to \infty$, or $p(\tau, N) \to 0$ as $N \to \infty$.

[20 marks]

5. Explain (without proofs) the method of diagrams and the method of complete diagrams.

Let $\mathcal{L}$ be a language and let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures with $\mathfrak{A} \subseteq \mathfrak{B}$. Assume that for all existential formulas $\phi(v_1, \ldots, v_n)$ of $\mathcal{L}$ and all $a_1, \ldots, a_n \in \text{dom}(\mathfrak{A})$,

$$\mathfrak{A} \models \phi[a_1, \ldots, a_n] \iff \mathfrak{B} \models \phi[a_1, \ldots, a_n].$$

Prove that there exists an $\mathcal{L}$-structure $\mathfrak{C}$ such that $\mathfrak{B} \subseteq \mathfrak{C}$ and $\mathfrak{A} \preceq \mathfrak{C}$.

[20 marks]