MATH63051 MODEL THEORY SPECIMEN SOLUTIONS and MARK SCHEME

[Marks out of 27, because of rubric.]

1) (a)(i) • v_i is a term for any $i \geq 1$.

• If τ is a term, so is $F(\tau)$.

(ii) $\tau_1 \simeq \tau_2$ for τ_1, τ_2 terms of $\mathcal{L}$.

(iii) • Every atomic fmla. is a fmla.

• If ϕ, ψ are fmlas, so are $\neg\phi$, $(\phi \wedge \psi)$, $\exists v_i \phi$ (for any $i \geq 1$).

(6 marks)

(b) • Say $\mathfrak{A} = \langle A; f \rangle$. Then $\pi: A \to A$ is an automorphism iff π is bijective and for all $a \in A$, $\pi(f(a)) = f(\pi(a))$.

• We must show that for any formula $\phi(\bar{v})$ of $\mathcal{L}$, and any $\bar{a} \subset A$, $\mathfrak{A} \models \phi[\bar{a}] \Leftrightarrow \mathfrak{A} \models \phi[\pi(\bar{a})]$. ---(*).

The proof is by induction on ϕ:

Firstly we must show that $\pi(\tau^{\mathfrak{A}}(\bar{a})) = \tau^{\mathfrak{A}}(\pi(\bar{a}))$ ~~for~~ --(†) for every term τ. This is by induction on τ:- if τ is v_i, then $\tau^{\mathfrak{A}}(\bar{a}) := a_i$ and $\tau^{\mathfrak{A}}(\pi(\bar{a})) := \pi(a_i)$, so result follows.

~~Final~~ If τ is $F(\tau_0)$ and result true for τ_0, we get

(α) $\tau^{\mathfrak{A}}(\bar{a}) := f(\tau_0^{\mathfrak{A}}(\bar{a}))$ and (β) $\tau^{\mathfrak{A}}(\pi(\bar{a})) := f(\tau_0^{\mathfrak{A}}(\pi(\bar{a})))$.

But by ind. hyp. $\pi(\tau_0^{\mathfrak{A}}(\bar{a})) = \tau_0^{\mathfrak{A}}(\pi(\bar{a}))$, so (from ($\beta$)) $\tau^{\mathfrak{A}}(\pi(\bar{a})) = f(\pi(\tau_0^{\mathfrak{A}}(\bar{a})))$

$= \pi(f(\tau_0^{\mathfrak{A}}(\bar{a})))$ (since π is an autom.) $= \pi(\tau^{\mathfrak{A}}(\bar{a}))$ (from (α)), as required.

Thus for any terms τ_1, τ_2, $\mathfrak{A} \models \tau_1 \simeq \tau_2[\bar{a}] \Leftrightarrow \tau_1^{\mathfrak{A}}(\bar{a}) = \tau_2^{\mathfrak{A}}(\bar{a})$ (def. of "$\models$")

$\Longleftrightarrow \pi(\tau_1^{\mathfrak{A}}(\bar{a})) = \pi(\tau_2^{\mathfrak{A}}(\bar{a}))$ (as π 1-1)

$\Longleftrightarrow \tau_1^{\mathfrak{A}}(\pi(\bar{a})) = \tau_2^{\mathfrak{A}}(\pi(\bar{a}))$ (by (†))

$\Longleftrightarrow \mathfrak{A} \models \tau_1 \simeq \tau_2[\pi(\bar{a})]$ (def of "$\models$").

So (*) is true for atomic ϕ.

Also $\mathfrak{A} \models \neg\phi[\bar{a}] \Longleftrightarrow$ not $\mathfrak{A} \models \phi[\bar{a}]$ (def of "$\models$")

$\Longleftrightarrow$ not $\mathfrak{A} \models \phi[\pi(\bar{a})]$ (ind hyp.)

$\Longleftrightarrow \mathfrak{A} \models \neg\phi[\pi(\bar{a})]$ (def of "$\models$");

And $\mathfrak{A} \models (\phi \wedge \psi)[\bar{a}] \Longleftrightarrow \mathfrak{A} \models \phi[\bar{a}]$ and $\mathfrak{A} \models \psi[\bar{a}]$ (def of "$\models$")

$\Longleftrightarrow \mathfrak{A} \models \phi[\pi(\bar{a})]$ and $\mathfrak{A} \models \psi[\pi(\bar{a})]$ (ind. hyp)

$\Longleftrightarrow \mathfrak{A} \models (\phi \wedge \psi)[\bar{a}]$ (def of "$\models$").

Finally, $\mathcal{C} \models \exists v_i \phi[\bar{a}] \iff$ for some $b_i \in A$, $\mathcal{C} \models \phi[a_1, a_{i-1}, b_i, a_{i+1}, \ldots, a_n]$

$\iff$ for some $b_i \in A$, $\mathcal{C} \models \phi[\pi(a_1), \ldots, \pi(a_{i-1}), \pi(b_i), \pi(a_{i+1}), \ldots, \pi(a_n)]$

(ind hyp.)

$\iff$ for some $c_i \in A$, $\mathcal{C} \models \phi[\pi(a_1), \ldots, \pi(a_{i-1}), c_i, \pi(a_{i+1}), \ldots, \pi(a_n)]$.

(since π is surjective).

$\iff$ $\mathcal{C} \models \exists v_i \phi[\pi(a_1), \ldots, \pi(a_n)]$ (def of "$\models$").

(10 marks)

(c) • $\Leftarrow$: $\neg v_1 \doteq v_1$ defines $\emptyset$ and $v_1 \doteq v_1$ defines $\mathbb{Z}$.

$\Rightarrow$: Suppose $S \subseteq \mathbb{Z}$ is $\mathcal{C}_0$-definable and $S \neq \emptyset$. Let $a \in S$.
Let $b \in \mathbb{Z}$ and consider the map $\pi : \mathbb{Z} \to \mathbb{Z} : x \mapsto x + (b-a)$.
Clearly π is bijective, and for all $x \in \mathbb{Z}$,
$\pi(s_0(x)) = s_0(x) + (b-a) = x + 1 + (b-a) = (x + (b-a)) + 1 = s_0(\pi(x))$.
Hence π is an automorphism of $\mathcal{C}_0$. Now let $\phi(v_1)$ be a formula defining S. Then $\mathcal{C}_0 \models \phi[a]$, so $\mathcal{C}_0 \models \phi[\pi(a)]$ by (b). Thus $\mathcal{C}_0 \models \phi[a + (b-a)]$, i.e. $\mathcal{C}_0 \models \phi[b]$, so $b \in S$.
Since $b \in \mathbb{Z}$ was arbitrary, $S = \mathbb{Z}$. (6 marks)

• We prove by induction on $n \in \mathbb{N}$, that $\{n\}$ is definable in $\mathcal{C}_1$.

– $n = 0$: Let $\phi_0(v_1)$ be $\forall v_2 \neg F(v_2) \doteq v_1$.

– Assume $\phi_n(v_1)$ defines $\{n\}$. Then $\exists v_p (\phi_n(v_p) \wedge v_1 \doteq F(v_p))$ defines $\{n+1\}$ (where p is chosen so that v_p is freely substitutable for v_1 in $\phi_n(v_1)$).

Now if $S = \{a_1, \ldots, a_r\} \underset{\text{fin}}{\subseteq} \mathbb{N}$, $(\phi_{a_1}(v_1) \vee \cdots \vee \phi_{a_r}(v_1))$ defines S in $\mathcal{C}_0$

(5 marks)

Remarks. (a) is bookwork, but they need to have some confidence not to churn out the definition for arbitrary $\mathcal{L}$. (b) was done in mid-term test. (c) is new but several examples have been gone through in problem classes, e.g. with automorphisms of $\langle \mathbb{R}; +; 0 \rangle$ and definability of the primes in $\langle \mathbb{N}; +, \cdot \rangle$.

2) • $\mathfrak{A} \preceq \mathfrak{B} \iff$ for all formulas $\phi(\bar{v})$ of $\mathcal{L}$ and all $\bar{a} \subset \mathrm{dom}(\mathfrak{A})$, we have $\mathfrak{A} \models \phi[\bar{a}] \iff \mathfrak{B} \models \phi[\bar{a}]$ (and $\mathrm{dom}(\mathfrak{A})$ is a subset of $\mathrm{dom}(\mathfrak{B})$).

(2 marks)

• Tarski's Lemma: $\mathfrak{A} \preceq \mathfrak{B}$ if and only if $\mathfrak{A} \subseteq \mathfrak{B}$ and whenever $a_1, \ldots, a_{p-1}, a_{p+1}, \ldots, a_n \in A$, $b_p \in B$ and $\phi(v_1, \ldots, v_n)$ is a formula of $\mathcal{L}$ with $\mathfrak{B} \models \phi[a_1, \ldots, a_{p-1}, b_p, a_{p+1}, \ldots, a_n]$, then there is some $a_p \in A$ with $\mathfrak{B} \models \phi[a_1, \ldots, a_{p-1}, a_p, a_{p+1}, \ldots, a_n]$.

(4 marks)

• Let $\mathfrak{A} = \langle A; < \rangle$, $\mathfrak{B} = \langle B; < \rangle$ where we use the same "$<$" since $\mathfrak{A} \subseteq \mathfrak{B}$. We have $A \subseteq B$ and A, B both countable. Now with notation as in ~~Tarski's Lemma, choose a~~ Tarski's Lemma, choose $a_p \in A$ having the same order relation to $a_1, \ldots, a_{p-1}, a_{p+1}, \ldots, a_n$ as b_p does.

(∗) --- E.g. by permuting ~~free variables~~ free variables we may as well suppose that $a_1 < \cdots < a_{p-1} < b_p < a_{p+1} < \cdots < a_n$. Then, if $p \geq 2$, a_p exists by the density of A. If $p = 1$, a_p exists since A has no least element, and if $p = n$ then a_p exists since A has no greatest element. --- (∗)

We now construct an automorphism, π, of $\mathfrak{B}$ such that $\pi(a_i) = a_i$ for $i \neq p$ $(1 \leq i \leq n)$, and $\pi(b_p) = a_p$. Since $\mathfrak{B} \models \phi[a_1, \ldots, a_{p-1}, b_p, a_{p+1}, \ldots, a_n]$ it follows that $\mathfrak{B} \models \phi[\pi(a_1), \ldots, \pi(b_p), \ldots, \pi(a_n)]$ (by preservation of truth given in question), i.e. $\mathfrak{B} \models \phi[a_1, \ldots, a_{p-1}, a_p, \ldots, a_n]$. So conclusion of Tarski's Lemma holds (as $a_p \in A$) and we are done.

To construct π, let $\{c_i : i \geq 1\}$ be an enumeration of the (countable) set B. Assume $c_1, \ldots, c_n = a_1, \ldots, a_{p-1}, b_p, a_{p+1}, \ldots, a_n$. ~~Suppose~~ Assume that $\pi(c_1), \ldots, \pi(c_i)$ have been defined for some $i \geq n$. Let $\pi(c_{i+1})$ have same $<$-relation to $\pi(c_1), \ldots, \pi(c_i)$ as c_{i+1} does to $c_1, \ldots, c_i$ (by same argument as (∗)). ~~This completes the construction, and $\pi: \mathfrak{B} \to \mathfrak{B}$ is~~ Further, choose $\pi(c_{i+1})$ to be the candidate c_j with j minimal.

This completes the construction, and clearly $\pi : B \to B$ is an embedding from $\mathcal{B}$ to $\mathcal{B}$. However, it is also surjective: just consider the least j s.th. $c_j \notin \mathrm{ran}(\pi)$ and choose N s.th. $c_1, \ldots, c_{j-1} \in \{\pi(c_1), \ldots, \pi(c_N)\}$. Eventually we will come across a c_p (and take p minimal) having same $<$-relation to $c_1, \ldots, c_N$ as c_j does to $\pi(c_1), \ldots, \pi(c_N)$, and the rule forces $\pi(c_p) = c_j$. Thus π <u>is</u> an automorphism of $\mathcal{B}$.

(14 marks)

- By downward L–S Theorem, let $\mathcal{B}$ be an elementary substructure of $\langle \mathbb{R}; < \rangle$, s.th (i) $\mathbb{Q} \subseteq \mathrm{dom}(\mathcal{B})$ and (ii) $\mathrm{dom}(\mathcal{B})$ is countable. Then $\langle \mathbb{Q}; < \rangle \subseteq \mathcal{B}$ and since $\mathcal{B} \equiv \langle \mathbb{R}; < \rangle$, $\mathcal{B}$ is a dense linear order without endpoints. Hence by above $\langle \mathbb{Q}; < \rangle \preccurlyeq \mathcal{B}$. So we have $\langle \mathbb{Q}; < \rangle \preccurlyeq \mathcal{B} \preccurlyeq \langle \mathbb{R}; < \rangle$, and so $\langle \mathbb{Q}; < \rangle \preccurlyeq \langle \mathbb{R}; < \rangle$.

(7 marks)

<u>Remarks</u> First two parts are bookwork. I have done the back-and-forth argument for dense linear orders in the lectures (and again in the proof of Ryll-Nardzewski), but not with the two structures being the same. I mention the fact that π is automatically surjective, but do the "back" part anyway, and I expect they will too. I have also done the use of automorphisms in connection with Tarski's Lemma (both as a general result, and with examples).

The last part is new.

3) (a) Let $F_n(T)$:= set of all formulas of $\mathcal{L}$ with free variables amongst $F_n(T)$.

(i) A *n-type* (over T) is a subset $p \subseteq F_n(T)$ s.th.

(1) for all $\phi, \psi \in p$, $(\phi \wedge \psi) \in p$

(2) for all $\phi \in p$, $T \models \exists v_1, .., v_n \phi$

(3) for all $\phi \in F_n(T)$, either $\phi \in p$ or $\neg\phi \in p$.

(ii) An n-type (over T) p is called *principal* if there is some $\phi \in p$ s.th. for all $\psi \in p$, $T \models \forall v_1, .., v_n (\phi \rightarrow \psi)$. (Then ϕ *generates* p.)

(4 marks)

(b) • Let $\mathfrak{A} \models T$, and let ϕ generate the principal n-type p.
By (2), $\mathfrak{A} \models \exists \bar{v} \phi$ (where $\bar{v} = v_1, .., v_n$) so choose $\bar{a} \in \mathrm{dom}(\mathfrak{A})^n$ s.th.
$\mathfrak{A} \models \phi[\bar{a}]$. Let $\psi \in p$. Then $\mathfrak{A} \models \forall \bar{v}(\phi \rightarrow \psi)$. In particular,
$\mathfrak{A} \models \psi[\bar{a}]$. Since $\psi \in p$ was arbitrary, $\bar{a}$ realises p in $\mathfrak{A}$.

(4 marks)

• Let p be any n-type (over T). Let $c_1, .., c_n$ be new constant symbols and let $\Sigma := \{\phi(c_1, .., c_n) : \phi \in p\}$. Since the conjunction of any finite subset of Σ is also in Σ (as follows immediately from (1) and induction), we see from (2) that Σ is finitely satisfiable. Hence Σ has a model, $\mathfrak{A}'$ say. Let $\mathfrak{A} = \mathfrak{A}' \upharpoonright \mathcal{L}$. Then $\langle c_1^{\mathfrak{A}'}, .., c_n^{\mathfrak{A}'} \rangle$ realises p in $\mathfrak{A}$.

(5 marks)

(c) OTT: Let p be a nonprincipal n-type (over T). Then there exists a model $\mathfrak{A} \models T$ such that $\mathfrak{A}$ omits (i.e. does not realise) p.

(1 mark)

(d) Let p be a nonprincipal n-type (over T). By (c) there exists a model $\mathfrak{A} \models T$ such that $\mathfrak{A}$ omits p. By dnd. L-S-theorem we may take a countable $\mathfrak{A}' \preccurlyeq \mathfrak{A}$. Then $\mathfrak{A}' \models T$ and clearly $\mathfrak{A}'$ also omits p. Now by (b), let $\mathfrak{B} \models T$ s.th. $\mathfrak{B}$ realises p, by $\bar{b}$ say. Again, by dns L-S-Theorem let $\mathfrak{B}' \preccurlyeq \mathfrak{B}$ with $\mathfrak{B}'$ countable and $\bar{b} \in \mathrm{dom}\, \mathfrak{B}'$. Then $\bar{b}$ realises p in $\mathfrak{B}'$ and $\mathfrak{B}' \models T$.
Clearly $\mathfrak{A}' \not\cong \mathfrak{B}'$, so T is not $\aleph_0$-categorical.

(4 marks)

(e) Let $\Sigma := \{\neg v_1^N \doteq c : N \geq 1\}$ where v_1^N denotes the term $\underbrace{v_1 \circ (v_1 \circ \cdots v_1)}_{N \text{ times}}$ and c is the constant for e. If Σ is not finitely satisfiable (over $Th(\mathcal{G})$), then for some $N_1, .., N_r \geq 1$ we have $\mathcal{G} \models \forall v_1 (v_1^{N_1} \doteq e \vee \dots \vee v_1^{N_r} \doteq e)$, which clearly implies $\mathcal{G} \models \forall v_1 (v_1^N \doteq e)$ where $N = N_1 \cdot N_2 \cdots N_r$. Thus for all $g \in \mathrm{dom}(\mathcal{G})$, $g^N = e$ as required.

So suppose Σ is fin. sat. Then we know (by Zorn's Lemma) that there exists a 1-type p (over $Th(\mathcal{G})$) with $p \supseteq \Sigma$. Suppose p were principal. Let $\phi(v_1) \in p$ generate p. By (a) we may choose $h \in \mathrm{dom}(\mathcal{G})$ such that $\mathcal{G} \models \phi[h]$. It now follows from the defn. of Σ ($\subseteq p$), that ~~[illegible]~~ $\mathcal{G} \models (\neg v_1^N \doteq e)[h]$ for all $N \geq 1$. Then $h^N \neq e$ for all $N \geq 1$, so h is an element of infinite order and we are done.

So p is a nonprincipal 1-type (over T). But then by (d) T is not $\aleph_0$ categorical. (9 marks)

<u>Remarks</u>: (a), (b), (c) are bookwork. So is (d), but they have to extract this easy part of the proof of Ryll-Nardzewski from the whole argument. They have seen the (stronger) version of (e) as a consequence of Ryll-Nardzewski. But here they have to get the weaker result from just (d), which they won't have seen before.

4) (i) First, the QF fmlas are defined inductively by:
- every atomic formula is QF;
- if ϕ, ψ are QF, then so are $\neg\phi$ and $(\phi \wedge \psi)$.

Then an existential fmla. is one of the form $\exists v_{i_1} \cdots \exists v_{i_r} \phi$ where ϕ is QF.

(ii) A universal sentence χ is a formula of the form $\forall v_{i_1} \cdots \forall v_{i_r} \phi$ where ϕ is QF and where there are no free occurrences of variables in χ. I.e. $FrVar(\phi) \subseteq \{v_{i_1}, \ldots, v_{i_r}\}$.

(5 marks)

~~(a) Suppose the statement~~

(a) • Let ϕ be $\exists v_{i_1} \cdots \exists v_{i_r} \psi$ with $FrVar(\phi) \subseteq \{v_1, \ldots, v_n\}$. Let $a_1, \ldots, a_n \in dom(\mathcal{A})$.

Then $\mathcal{A} \models \phi[a_1, \ldots, a_n]$ $\iff$ for some $b_{i_1} \in A, \ldots,$ for some $b_{i_r} \in A$, $\mathcal{A} \models \phi[a_1', \ldots, a_n']$ (by def of "$\models$").

$\left(\text{where } a_i' = \begin{cases} b_i & \text{if } i = i_j \text{ for some } j = 1, \ldots, r \\ a_i & \text{otherwise.} \end{cases}\right)$

$\Rightarrow$ for some $b_{i_1} \in B, \ldots,$ for some $b_{i_r} \in B$, $\mathcal{A} \models \phi[a_1', \ldots, a_n']$ (since $A \subseteq B$)

$\Rightarrow$ " " ", $\mathcal{B} \models \phi[a_1', \ldots, a_n']$ (since we are given that the statement is true for QF ϕ)

$\Rightarrow$ $\mathcal{B} \models \phi[a_1, \ldots, a_n]$ (by def. of "$\models$").

(5 marks)

• Let χ be a universal sentence. Then $\neg\chi$ is logically equivalent to an existential sentence. Hence if $\mathcal{A} \models \neg\chi$ then $\mathcal{B} \models \neg\chi$ (by above). Thus, $\mathcal{B} \models \chi$ implies $\mathcal{A} \models \chi$, as required

(2 marks)

(b) Let $A_0 = \{\tau^{\mathcal{A}} : \tau \text{ a closed term of } \mathcal{L}\}$. It is sufficient to show that $A_0 \neq \emptyset$ and that A_0 contains all distinguished elements of $\mathcal{A}$ and is closed under ~~the~~ $F^{\mathcal{A}}$ for each function symbol F of $\mathcal{L}$. (For then we interpret each relation symbol R of $\mathcal{L}$ by $A_0^s \cap R^{\mathcal{A}}$ (where s = arity of R), each function symbol F of $\mathcal{L}$ by $F^{\mathcal{A}} \restriction A_0^t$ (where t = arity of F), and each constant c of $\mathcal{L}$ by $c^{\mathcal{A}}$.)

The first two assertions are clear since each constant symbol of $\mathcal{L}$ is a closed term of $\mathcal{L}$, and we are given that there is at least one.

For the last assertion, let $a_1, \dots, a_t \in A_0$, say $a_j = \tau_j^{\mathfrak{A}}$ $(1 \le j \le t)$, and let F be a t-ary function symbol of $\mathcal{L}$. Let τ be the closed term $F(\tau_1, \dots, \tau_t)$. Then $\tau^{\mathfrak{A}} \in A_0$. However, by defn. of interpretation of terms, $\tau^{\mathfrak{A}} = F^{\mathfrak{A}}(\tau_1^{\mathfrak{A}}, \dots, \tau_t^{\mathfrak{A}})$

$= F^{\mathfrak{A}}(a_1, \dots, a_t)$, and so $F^{\mathfrak{A}}(a_1, \dots, a_t) \in A_0$ and we are done.

(7 marks)

(c) Suppose false.

Let $\Sigma := T \cup \{\neg \phi(\tau) : \tau$ a closed term of $\mathcal{L}\}$.

Then Σ is finitely satisfiable, for otherwise there exist $\tau_1, \dots, \tau_N$ (closed terms) such that $T \cup \{\neg\phi(\tau_1), \dots, \neg\phi(\tau_N)\}$ is unsatisfiable. Thus $T \models (\phi(\tau_1) \vee \dots \vee \phi(\tau_N))$ — contradiction.

So Σ has a model, $\mathfrak{A}$ say, by the Compactness Theorem. Let $\mathfrak{A}_0 \subseteq \mathfrak{A}$ be as in (b). Now since T consists of universal sentences and $\mathfrak{A} \models T$, we have $\mathfrak{A}_0 \models T$ (by second part of (a)). ~~Then~~ Hence $\mathfrak{A}_0 \models \exists v_1 \phi(v_1)$ (as $T \models \exists v_1 \phi(v_1)$). Say $a \in A_0$ and $\mathfrak{A}_0 \models \phi[a]$. Choose a closed term τ s.th. $a = \tau^{\mathfrak{A}}$, so that $\mathfrak{A}_0 \models \phi[\tau^{\mathfrak{A}}]$. ~~By standard (given) facts, $\mathfrak{A}_0 \models \phi(\tau)$. But $\phi(\tau)$ is an existential sentence, so by first part of (a)~~ By first part of (a) (since $\phi(v_1)$ is existential) we get $\mathfrak{A} \models \phi[\tau^{\mathfrak{A}}]$. By standard (given) facts, $\mathfrak{A} \models \phi(\tau)$. However, $\neg\phi(\tau) \in \Sigma$ and $\mathfrak{A} \models \Sigma$ — contradiction.

(8 marks)

Remarks. First part, and (a) is bookwork. So is (b), but it was done in the context of the dnd. LS-theorem and the Compactness Theorem, so they will have to spot that. (c) is new, but much harder proofs along these lines were done in the lectures in the context of preservation theorems.