1) (a) $v_i$ is a term for any $i \geq 1$.
(ii) $z_i = z_j$ for $z_i, z_j$ terms of $L$.
(iii) Every atomic formula is a formula.

If $\phi, \psi$ are formulas, so are $\neg \phi$, $(\phi \psi)$, $\forall x \phi$ (for any $i \geq 1$).

(b) Say $C \alpha = \langle A; \delta \rangle$. Then $\pi : A \to A$ is an automorphism iff $\pi$ is bijective and for all $a \in A$, $\pi(\delta(a)) = \delta(\pi(a))$.

We must show that for any formula $\phi(\bar{v})$ of $L$, and any $\bar{v} \in A^n$, $C \alpha \vdash \phi[\bar{v}] \iff C \alpha \vdash \phi[\pi(\bar{v})]$.

The proof is by induction on $\phi$:

Firstly we must show that $\pi(\zeta^C(\alpha)) = \zeta^C(\pi(\alpha))$ for every term $\zeta$.

This is by induction on $\zeta$:

If $\zeta$ is a constant, then $\zeta^C(\alpha) = \alpha$ and $\zeta^C(\pi(\alpha)) = \pi(\alpha)$, so result follows.

If $\zeta \in F(C_0)$ and result true for $C_0$, we get $\zeta^C(\alpha) = \phi(\zeta)$.

Proof by induction.

For any $\alpha, \beta, \gamma$, $\pi(\zeta^C(\alpha)) = \zeta^C(\pi(\alpha))$.

Thus for any terms $z_1, z_2$, $C \alpha \vdash z_1 \equiv z_2 [\alpha] \iff \zeta^C(\alpha) = \zeta^C(\pi(\alpha))$ (def of $\equiv$).

$(*)$\quad $C \alpha$ is true for atomic $\phi$.

Also $C \alpha \vdash \neg \phi[\bar{v}] \iff$ not $C \alpha \vdash \phi[\bar{v}]$ (def of $\vdash$).

And $C \alpha \vdash (\phi \psi)[\bar{v}]$ $\iff$ $C \alpha \vdash \phi[\bar{v}]$ and $C \alpha \vdash \psi[\bar{v}]$ (def of $\vdash$).
Finally, \( C_2 \models \exists x \phi \[ x \] \iff \text{for some } b \in A, \ C_2 \models \phi \[ a_0, a_1, \ldots, b, a_{m}, \ldots, a_n \] \\
\iff \text{for some } b \in A, \ C_2 \models \phi \[ \pi(a_0), \ldots, \pi(a_{m}), \pi(b), \pi(a_{m+1}), \ldots, \pi(a_n) \] \\
\iff \text{for some } c \in A, \ C_2 \models \phi \[ \pi(a_0), \ldots, \pi(a_{m}), c, \pi(a_{m+1}), \ldots, \pi(a_n) \] \\
\iff \text{for some } c \in A, \ C_2 \models \phi \[ \pi(a_0), \ldots, \pi(a_{m}), c, \pi(a_{m+1}), \ldots, \pi(a_n) \]. \\
(\text{since } \pi \text{ is surjective}) \\
\iff \ C_2 \models \exists v_i \phi \[ \pi(a_0), \ldots, \pi(a_{m}) \] \text{ (def of } \forall \text{)}. \\
(10 \text{ marks})
\[
\left( v_i \right) c \iff \neg v_i \equiv v_i \text{ defines } 0 \text{ and } v_i \equiv v_i \text{ defines } \mathbb{Z}.
\]

\[ \Rightarrow \quad \text{Suppose } S \subseteq \mathbb{Z} \text{ is } A_0 \text{-definable and } S \neq \emptyset. \text{ Let } a \in S. \]

Let \( b \in \mathbb{Z} \) and consider the map \( \pi : \mathbb{Z} \rightarrow \mathbb{Z} : x \mapsto x + (b-a). \)

Clearly \( \pi \) is bijective, and for all \( x \in \mathbb{Z} \),
\[ \pi(x) = x + (b-a) = (x+b-a) = 0_0(\pi(x)). \]

Hence \( \pi \) is an automorphism of \( A_0 \). Now let \( \phi(v_i) \) be

a formula defining \( S \). Then \( A_0 \models \phi \[ a \] \), so \( A_0 \models \phi \[ \pi(a) \] \\
by (\ell). Then \( A_0 \models \phi \[ a+(b-a) \] \), i.e. \( A_0 \models \phi \[ b \] \), so \( b \in S \).

Since \( b \in \mathbb{Z} \) was arbitrary, \( S = \mathbb{Z} \). \\
(6 marks)

So we prove by induction on \( n \in \mathbb{N} \), that \( \exists x^n \) is definable in \( A_0 \).

\[ \begin{array}{|l|}
\hline
n=0: & \exists x_0 \phi_0(v_i) \text{ be } \forall v_{x_0} \rightarrow F(v_i) \equiv F(v_i) \\
\hline
\end{array} \]

Assume \( \phi_n(v_i) \) defines \( \exists x^n \). Then \( \exists v_0 (\phi_0(v_0) \land v_i \equiv F(v_0)) \)

defines \( \exists x^{n+1} \) (where \( v_0 \) is chosen so that \( v_0 \) is freely substituted for \( v_i \) in \( \phi_n(v_i) \)).

Now if \( S = \{ a_1, \ldots, a_m \} \subseteq \mathbb{N} \), \( \phi_n(v_i) \land \ldots \land \phi_n(v_i) \) defines \( S \)

in \( A_0 \) \\
(5 marks)

Remarks. (a) is bookwork, but they need to have some confidence. Not to churn out the definition for arbitrary \( \mathcal{L} \). (b) was done in mid-term test. (c) is new but several examples have been gone through in the problem classes, e.g. with automorphisms of \( \langle \mathbb{R}, +, \circ \rangle \) and definability of the primes in \( \langle \mathbb{N}, +, \cdot \rangle \).
2) \( \phi \leq \psi \iff \) for all formulas \( \phi(x) \) of \( L \) and all \( \alpha \in \text{dom}(\alpha) \),
we have \( \phi[\alpha] \iff \psi[\alpha] \) (and \( \text{dom}(\alpha) \) is a subset of \( \text{dom}(\beta) \)).

(2 marks)

**Tanaka's Lemma:** \( \phi \leq \psi \) if and only if \( \phi \leq \psi \) and
whenever \( a_1, \ldots, a_n \in A \), \( b_1 \in B \), \( \phi(v_1, \ldots, v_n) \) is
a formula of \( L \) with \( \phi[a_1, \ldots, a_n] = \phi[b_1, \ldots, a_n] \), then
there is some \( b_1 \in B \) with \( \phi[a_1, \ldots, a_n, b_1] = \phi[a_1, \ldots, a_n] \).

(4 marks)

Let \( \varphi \equiv \langle A; \prec \rangle \), \( \psi \equiv \langle B; \prec \rangle \) where we use the same
"\( \prec \)" since \( A \subseteq B \). We have \( A \subseteq B \) and \( A, B \) both countable.
Now with notation as in Tanaka's Lemma, choose \( a_0 \in A \) having the
same order relation to \( a_1, a_2, \ldots, a_n \) as \( b_0 \) does.

(4 marks)

\( \ast \) Example: By permuting \( \phi \) without free variables we may as well
suppose that \( a_1 < \cdots < a_n < b_0 < a_{n+1} < \cdots < a_m \). Then,
if \( n \geq 2 \), there exists by the density of \( A \).
If \( n = 1 \), \( a_0 \) exists since \( A \) has no least element, and if \( n = 2 \), \( a_0 \) exists
since \( A \) has no greatest element.

(\( \ast \))

We now construct an automorphism \( \pi \) of \( B \) such that
\( \pi(a_0) = a_2 \) for \( i \neq 0 \) \((1 \leq i \leq n)\), and \( \pi(a_0) = a_0 \).

Since
\( \phi \equiv \phi[a_1, \ldots, a_n, b_0, a_{n+1}, \ldots, a_m] \) it follows that \( \phi[\pi(a_1), \ldots, \pi(a_n), \pi(a_{n+1}), \ldots, \pi(a_m)] \)
(by preservation of truth given in question), i.e.
\( \phi[\pi(a_1), \ldots, a_n, \pi(a_{n+1}), \ldots, \pi(a_m)] \).
So conclusion of Tanaka's Lemma holds \((\text{as} a_0 \in A)\) and we are done.

To construct \( \pi \), let \( \varphi \equiv \langle c_i; i > \rangle \) be an enumeration of the
(countable) set \( B \). Assume \( c_1, \ldots, c_n = a_1, \ldots, a_n, b_1, a_{n+1}, \ldots, a_m \).
Suppose \( \pi(c_i) \) have been defined for \( \pi(c_1), \ldots, \pi(c_i) \).
In \( \pi(c_{i+1}) \) have same < relation to \( \pi(c_1), \ldots, \pi(c_i) \) as \( c_{i+1} \) does
to \( c_1, \ldots, c_i \) (by same argument as \( \ast \)). Further, choose \( \pi(c_{i+1}) \)
to be the candidate \( c_j \) with \( j \) minimal.
This completes the construction, and clearly \( \pi : B \to B \) is an embedding from \( L_\theta \) to \( L_\theta \). However, it is also surprising: just consider the least \( j \) s.th. \( \xi_j \notin \text{ran}(\pi) \) and choose \( N \) s.th. \( \xi_1, ..., \xi_{j-1} \notin \pi(\xi_1), ..., \pi(\xi_N) \). Eventually we will come across a \( \xi_p \) (and take \( p \) minimal) having some \( \prec \)-relation to \( \xi_1, ..., \xi_N \) as \( \xi_j \) does to \( \pi(\xi_1), ..., \pi(\xi_N) \), and the rule forces \( \pi(\xi_p) = \xi_j \). Thus \( \pi \) is an automorphism of \( B \).

(14 marks)

By downward L-S Theorem, let \( B \) be an elementary substructure of \( \langle \mathbb{R}; < \rangle \), s.th (i) \( B \subseteq \text{dom}(\beta) \) and (ii) \( \text{dom}(\beta) \) is countable.

Then \( \langle \mathbb{Q}; < \rangle \subseteq B \) and since \( \mathbb{Q} \equiv \langle \mathbb{R}; < \rangle \), \( B \) is a dense linear order without endpoints. Hence by above \( \langle \mathbb{Q}; < \rangle \subseteq B \).

So we have \( \langle \mathbb{Q}; < \rangle \subseteq B \subseteq \langle \mathbb{R}; < \rangle \), and \( \langle \mathbb{Q}; < \rangle \subseteq \langle \mathbb{R}; < \rangle \).

(7 marks)

Remarks. First two parts are bookwork. I have done the back-and-forth argument for dense linear order in the lectures (and again in the proof of Ryll-Nardzewski), but not with the true structures being the same. I mention the fact that \( \pi \) is automatically surjective, but do the "back" part anyway, and I expect they will too. I have also done the use of automorphisms in connection with Tarski's Lemma (both as a general result, and with examples).

The last part is new.
3) (a) Let $F_n(T)$ denote the set of all formulas of $L$ with free variables amongst $F_n(T)$.

(i) A $n$-type $(\phi(T))$ in a structure $\mathcal{M}$ is a subset $\rho \subseteq F_n(T)$ s.t.

1) for all $\phi, \psi \in \rho$, $(\phi \lor \psi) \in \rho$

2) for all $\phi \in \rho$, $T \models \forall \phi_1, \ldots, \phi_n \phi$

3) for all $\phi \in F_n(T)$, either $\phi \in \rho$ or $\neg \phi \in \rho$.

(ii) An $n$-type $(\phi(T))$ is called principal if there is some $\phi \in \rho$ with, for all $\psi \in \rho$, $T \models \forall \psi_1, \ldots, \psi_n (\phi \rightarrow \psi)$. (Then $\phi$ generates $\rho$.)

(b) Let $\mathcal{G} \models T$, and let $\phi$ generate the principal $n$-type $\rho$.

By (2), $\mathcal{G} \models \forall \phi$ (where $\phi = \psi_1, \ldots, \psi_n$) so choose $\bar{a} \in \text{dom}(\mathcal{G})^n$ s.t. $\mathcal{G} \models \phi[\bar{a}]$. Let $\psi \in \rho$. Then $\mathcal{G} \models \forall \psi (\phi \rightarrow \psi)$. In particular, $\mathcal{G} \models \phi[\bar{a}]$. Since $\psi \in \rho$ was arbitrary, $\bar{a}$ realises $\rho$ in $\mathcal{G}$.

(c) Let $\rho$ be any $n$-type (over $T$). Let $c_1, \ldots, c_n$ be the new constant symbols and let $\mathcal{E} := \{ \phi(c_1, \ldots, c_n) : \phi \in \rho \}$. Since the conjunction of any finite subset of $\mathcal{E}$ is also in $\mathcal{E}$ (as follows immediately from (i) and induction), we see from (2) that $\mathcal{E}$ is finitely satisfiable. Hence $\mathcal{E}$ has a model, $\mathcal{G}'$, say, with $\mathcal{G} \models \mathcal{G}'[\bar{a}]$. Then $\langle c_1, \ldots, c_n \rangle$ realises $\rho$ in $\mathcal{G}'$.

(c) OTT: Let $\rho$ be a nonprincipal $n$-type (over $T$). Then there exists a model $\mathcal{G} \models T$ such that $\mathcal{G}$ omits $\rho$ (i.e. does not realise $\rho$).

(d) Let $\rho$ be a nonprincipal $n$-type (over $T$). By (c) there exists a model $\mathcal{G} \models T$ such that $\mathcal{G}$ omits $\rho$. By and L-S Theorem we may like a countable $\mathcal{G}' \subseteq \mathcal{G}$. Then $\mathcal{G}' \models T$ and clearly $\mathcal{G}'$ also omits $\rho$. Now by (e), let $\mathcal{G} \models T$ s.t. $\mathcal{G}$ realises $\rho$, by $\mathcal{G}'$ say. Again, by and L-S Theorem let $\mathcal{G}' \models T$ with $\mathcal{G}'$ countable and $\mathcal{G} \models T$. Then $\mathcal{G}'$ realises $\rho$ in $\mathcal{G}'$ and $\mathcal{G} \models T$.

Clearly $\mathcal{G}' \models T$, so $T$ is not $\aleph_0$-categorical.

(e) Let $\mathcal{E} := \{ \neg v_1^{\mathcal{M}} = \bar{c} \ : \ N \gg 1 \}$, where $v_1^{\mathcal{M}}$ denotes the term $v_1(v_0, \ldots, v_i)$ and $\bar{c}$ the constant $\bar{c}$ for $e$. If $\mathcal{E}$ is not

finitely satisfiable (over $\mathcal{G}(\mathcal{G})$), then for some $\mathcal{G}_1, \ldots, \mathcal{G}_n \gg 1$ we
\[ \forall y \forall v_i (v_i^N = x \lor \cdots \lor v_i^{N_r} = c), \] which clearly
implies
\[ \forall y \forall v_i (v_i^N = x) \quad \text{where} \quad N = N_1 \cdot N_2 \cdots N_r. \] Thus for
all \( q \in \text{dom}(\lambda y), \) \( g^N = e \) as required.

So suppose \( E \) is fin. srt. Then we know (by Zorn's lemma) that there exists a 1-type \( p \) (over \( \text{Th}(x) \)) with \( p \supset E \). Suppose \( p \) were principal. Let \( \phi (v_i) \in p \) generate \( p \). By (2) we may
choose \( g \in \text{dom}(\lambda y) \) such that \( y^N = \phi^N \). It now follows from the
defn. of \( E \) (\( \supset p \)), that \( \forall y \exists N \exists \alpha \exists \beta \quad y^N = (\forall v_i^N = c)^N \) for all \( N \geq 1 \).

Then \( g^N = e \) for all \( N \geq 1 \), so \( h^N \) is an element of infinite order
and we are done.

So \( p \) is a nonprincipal 1-type (over \( T \)). But then by (1), \( T \)
is not \( \omega \)-categorical.

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Remarks: (a), (d), (c) are bookwork. So is (d), but they have to
exist. This easy part of the proof of Ryll-Nardzewski from
the whole argument. They have seen the (stronger) version of (e)
as a consequence of Ryll-Nardzewski. But here they have to
get the weaker result from just (d), which they won't have
seen before.
4) (i) First, the AF formulas are defined inductively by:
- every atomic formula is AF;
- if \( \phi, \psi \) are AF, then so are \( \neg \phi \) and \( \phi \lor \psi \).

Then an existential formula is one of the form \( \exists v_1, \ldots, \exists v_n \phi \) where \( \phi \) is AF.

(ii) A universal sentence \( \forall X \) is a formula of the form \( \forall v_1, \ldots, \forall v_n \phi \) where \( \phi \) is AF and when there are no free occurrences of variables in \( X \). i.e. \( \text{TrVar}(\phi) \subseteq \{v_1, \ldots, v_n\} \).

(a) Let \( \phi \) be \( \exists v_1, \ldots, \exists v_n \psi \) with \( \text{TrVar}(\phi) \subseteq \{v_1, \ldots, v_n\} \).

Then \( \text{AF}(\phi[a_1, \ldots, a_n]) \iff \) for some \( b_i \in A, \ldots, \) for some \( b_i \in A, \text{AF}(\phi[b_1, \ldots, b_n]) \) (where \( a_i' = \{ \begin{array}{ll} b_i & \text{if \, some \, } i = j \text{ \, for some } j = 1, \ldots, r \text{,} \\
\text{otherw.} & \text{,} \end{array} \) (typo of "= ").

\[ \Rightarrow \text{for some } b_i \in B, \ldots, \text{for some } b_i \in B, \text{AF}(\phi[b_1, a_i', \ldots, a_n']) \text{ (since } A \subseteq B) \]

\[ \Rightarrow \text{for some } b_i \in B, \ldots, \text{for some } b_i \in B, \text{AF}(\phi[b_1, a_i', \ldots, a_n']) \text{ (since we are given that the statement is true for } \text{AF}(\phi) \text{)} \]

\[ \Rightarrow \text{AF}(\phi[a_1, \ldots, a_n']) \text{ (by def. of } \langle = \rangle \text{)} \)

(b) Let \( A_0 = \{ X \in \mathcal{L} : \text{ } X \text{ a closed term of } \mathcal{L} \} \). It is sufficient to show that \( A_0 \neq \emptyset \) and that \( A_0 \) contains all distinguished elements of \( \mathcal{C} \) and is closed under \( \text{TrVar} \) for each function symbol \( F \) of \( \mathcal{L} \). (For then we interpret each relation symbol \( R \) of \( \mathcal{L} \) by \( A_0 \times A_0 \) (where \( s = \text{arity of } R \)), each function symbol \( F \) of \( \mathcal{L} \) by \( A_0^r \) (where \( t = \text{arity of } F \)), and each constant \( c \) of \( \mathcal{L} \) by \( \{c\} \).

The first two assertions are clear since each constant symbol of \( \mathcal{L} \) is a closed term of \( \mathcal{L} \), and we are given that there is at least one.
For the last assertion, let $a_1, \ldots, a_n \in A_0$, say $a_j = \tau^a_j (1) \in A_0$, and let $F$ be a $r$-ary function symbol of $L$. Let $\tau^a$ be the closed term $F(\tau^{a_1}, \ldots, \tau^{a_n})$. Then $\tau^{a}\in A_0$. However, by definition of interpretation of terms, $\tau^{a} = F^c(a_1, \ldots, a_n)$, and so $F^{c}(a_1, \ldots, a_n) \in A_0$, and we are done.

(c) Suppose false.

Let $S := \{ \phi(z) : \text{a closed term of } L \}$. Then $S$ is finitely satisfiable, for otherwise there exist $\tau_1, \ldots, \tau_n$ (closed terms) such that $T \cup \{\phi(\tau_1), \ldots, \phi(\tau_n)\}$ is unsatisfiable. Then $T \models (\phi(\tau_1) \lor \ldots \lor \phi(\tau_n))$ — contradiction.

So $S$ has a model, $C_0$, say, by the Compactness Theorem. Let $C_0 \subseteq C_0$ be as in (b). Now since $T$ consists of universal sentences and $C \models T$, we have $C_0 \models T$ (by second part of (a)).

Then hence $C_0 \models \exists v_1 \phi(v_1)$ (as $T \models \exists v_1 \phi(v_1)$). Say $a \in A_0$ and $C_0 \models \phi[\tau^a]$. Choose a closed term $\tau$ s.t. $a = \tau^a$, so that $C_0 \models \phi[\tau^a]$. By standard (given) facts, $C_0 \models \phi(\tau)$. But $\phi(\tau)$ is a existential sentence, so by first part of (a) we get $C_0 \models \phi[\tau^a]$. By first part of (a) (since $\phi(\tau)$ is existential) we get $C_0 \models \phi(\tau^a)$. By standard (given) facts, $C \models \phi(\tau)$. However, $\neg \phi(\tau) \in S$ and $C \models S$ — contradiction.

Remarks. First part, and (a) in bookwork. So in (b), but it was done in the context of the dual LS Theorem and the Compactness Theorem, so they will have to spot that. (c) is new, but much harder proofs along these lines were done in the lectures in the context of preservation theorems.