Two and a half hours

THE UNIVERSITY OF MANCHESTER

MODEL THEORY

Date ???

Time ???

The total number of marks on the paper is 80. Each question carries equal weight and you may attempt as many questions as you like. Full marks on the paper may be obtained by the equivalent of complete answers to THREE questions. A further 20 marks are available from work during the semester, making a total of 100.

Electronic calculators may be used, provided that they cannot store text.

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1. Let $\mathcal{L}$ be the language containing just one unary function symbol F.

(a) Give precise definitions, in the case of this particular language, for (i) the set of *terms* of $\mathcal{L}$, (ii) the set of *atomic formulas* of $\mathcal{L}$, and (iii) the set of *formulas* of $\mathcal{L}$.

(b) Let $\mathfrak{A}$ be an $\mathcal{L}$-structure with $dom(\mathfrak{A})=A$. What does it mean to say that a map $\pi : A \to A$ is an *automorphism* of $\mathfrak{A}$? Prove that an automorphism preserves the truth of all formulas.

(c) Now let $\mathfrak{A}_0$, $\mathfrak{A}_1$ be the $\mathcal{L}$-structures $\langle \mathbb{Z}; s_0 \rangle$, $\langle \mathbb{N}; s_1 \rangle$ respectively, where s_o denotes the successor function on the set of integers $\mathbb{Z}$ (i.e. $s_0(x) = x + 1$) and s_1 the successor function on the set of nonnegative integers $\mathbb{N}$. Prove that a subset S of $\mathbb{Z}$ is definable in $\mathfrak{A}_0$ if and only if $S=\emptyset$ or $S=\mathbb{Z}$. Show, however, that every finite subset of $\mathbb{N}$ is definable in $\mathfrak{A}_1$.

2. Let $\mathcal{L}$ be a language and let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures. What does it mean to say that $\mathfrak{A}$ is an *elementary substructure* of $\mathfrak{B}$ (i.e. $\mathfrak{A} \preceq \mathfrak{B}$)? State *Tarski's Lemma* giving necessary and sufficient conditions for $\mathfrak{A}$ to be an elementary substructure of $\mathfrak{B}$.

Suppose now that $\mathcal{L}$ contains just a binary relation symbol, that $\mathfrak{A}$ and $\mathfrak{B}$ are both countable dense linear orders without endpoints and that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$. By sketching a suitable back-and-forth construction prove that $\mathfrak{A} \preceq \mathfrak{B}$. [You may assume that automorphisms preserve truth.]

Hence, by quoting a suitable Löwenheim-Skolem Theorem, prove that $\langle \mathbb{Q}; < \rangle \preceq \langle \mathbb{R}; < \rangle$.

3. Let $\mathcal{L}$ be a countable language and T a complete $\mathcal{L}$-theory without finite models.

(a) What is meant by (i) an *n-type (over T)*, and (ii) a *principal* n-type (over T)?

(b) Prove that a principal n-type (over T) is realised in all models of T and that any n-type (over T) is realised in some model of T.

(c) State the Omitting Types Theorem.

(d) Suppose that there exists a nonprincipal 1-type (over T). Prove that T is not $\aleph_0$-categorical.

(e) Let $\mathcal{G}$ be an infinite Abelian group whose complete theory is $\aleph_0$-categorical. Prove directly from (d) that either (1) there exists a positive integer N such that for all $g \in dom(\mathcal{G})$ we have that $g^N = e$, where e denotes the identity element of $\mathcal{G}$, or (2) there is some element $h \in dom(\mathcal{G})$ such that h has infinite order (i.e. for all $N \geq 1$, $h^N \neq 1$).

4. Let $\mathcal{L}$ be a language. What is meant by (i) an *existential* formula of $\mathcal{L}$ and (ii) a *universal sentence* of $\mathcal{L}$? [You may take as given the notion of an atomic formula of $\mathcal{L}$.]

(a) Prove that if $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{L}$-structures with $\mathfrak{A} \subseteq \mathfrak{B}$, then for all existential formulas $\phi(v_1, \ldots, v_n)$ of $\mathcal{L}$ and all $a_1, \ldots, a_n \in dom(\mathfrak{A})$, we have that $\mathfrak{A} \models \phi[a_1, \ldots, a_n]$ implies $\mathfrak{B} \models \phi[a_1, \ldots, a_n]$. [You may assume that this is true for formulas ϕ that contain no occurrences of quantifiers.] Deduce that any universal sentence that is true in $\mathfrak{B}$ is also true in $\mathfrak{A}$.

Assume now that $\mathcal{L}$ contains at least one constant symbol.

(b) Let $\mathfrak{A}$ be any $\mathcal{L}$-structure. Prove that there exists a substructure $\mathfrak{A}_o$ of $\mathfrak{A}$ such that for every $a \in dom(\mathfrak{A}_o)$ there exists a closed term τ of $\mathcal{L}$ such that $a = \tau^{\mathfrak{A}}$.

(c) Deduce that if T is a set of universal sentences of $\mathcal{L}$ and $\phi(v_1)$ is an existential formula of $\mathcal{L}$ such that $T \models \exists v_1 \phi(v_1)$, then there exist $N \geq 1$, and closed terms $\tau_1, \ldots, \tau_N$ of $\mathcal{L}$ such that $T \models (\phi(\tau_1) \vee \phi(\tau_2) \vee \ldots \vee \phi(\tau_N))$. [Hint: Assume false and use the Compactness Theorem.]