Three hours

UNIVERSITY OF MANCHESTER

FOURIER ANALYSIS AND LEBESGUE INTEGRATION

20th January 2012

9:45am – 12:45pm

Answer **ALL** of the seven questions in Section A and **THREE** of the four questions in Section B and **ALL** of Section C.
If you answer more the three questions in Section B then your best three solutions will be used.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** of the seven questions

A1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with period 2π.

(a) Write down a formula for the constant term in the *Fourier series* associated to f.

(b) Suppose that f and f' are piecewise continuous. Write down a formula for the value of the Fourier series associated to f at a point x where f is *not* continuous.

[4 marks]

A2. What is meant by saying that a set is *countable*?

[2 marks]

A3. What is meant by saying that a collection $\mathcal{B}$ of subsets of a set X is a *σ-algebra*?
Explain how the Borel σ-algebra of $\mathbb{R}$ is defined and show that it contains all closed intervals.
Give an example of a σ-algebra of subsets of $\mathbb{R}$ which does not contain all closed intervals. (You do not need to prove that your example is a σ-algebra.)

[5 marks]

A4. Let $\mathcal{A}$ be a σ-algebra of subsets of $\mathbb{R}$. What is meant by saying that a function $m : \mathcal{A} \to \mathbb{R}^+$ satisfies *translation invariance*?

[3 marks]

A5. For $i = 1, \ldots, k$, let α_i be a real number, and $E_i \subset \mathbb{R}$ be a Lebesgue measurable set. Assume that $E_i \cap E_j = \emptyset$ when $i \neq j$. Define $f : [0,1] \to \mathbb{R}$ by $f = \sum_{i=1}^{k} \alpha_i \chi_{E_i}$. Write down a formula for

$$\int f \, d\mu.$$

[3 marks]

A6. Is the following definition of the space $L^2([-\pi,\pi],\mu,\mathbb{R})$ correct?
$L^2([-\pi,\pi],\mu,\mathbb{R})$ is the space of all square integrable functions $f : [-\pi,\pi] \to \mathbb{R}^*$.
If not, give the correct definition.

[4 marks]

A7. What is meant by saying that a vector space V over $\mathbb{R}$, equipped with a map $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$, is an *inner product space*.

[4 marks]

SECTION B

Answer **THREE** of the four questions.

B8.

(i) State what it means for a set $E \subset \mathbb{R}$ to be a null set.

[2 marks]

(ii) Let $\{E_j\}_{j=1}^{\infty}$ be a countable collection of null sets. Show that $\bigcup_{j=1}^{\infty} E_j$ is a null set.

[7 marks]

Let $K \subset \mathbb{R}$ be the set obtained by the following modification of the construction of the Middle Third Cantor set. Let $K_0 = [0, 1]$. Obtain a new set K_1 by removing the open middle quarter, $(3/8, 5/8)$, of K_0, so that $K_1 = [0, 3/8] \cup [5/8, 1]$. Repeat this procedure to obtain sets $K_2, K_3, \ldots$ and define

$$K = \bigcap_{n=0}^{\infty} K_n.$$

(iii) How many disjoint intervals make up K_n? What is their length?

[2 marks]

(iv) Show that K is a null set.

[10 marks]

(v) Explain why K is a Borel set.

[4 marks]

B9.

(i) What is meant by saying that a function $f : [0,1] \to \mathbb{R}^*$ is *measurable*?

[3 marks]

(ii) What is meant by saying that two functions $f, g : [0,1] \to \mathbb{R}^*$ are equal μ*-almost everywhere*?

[2 marks]

(iii) Suppose that $f : [0,1] \to \mathbb{R}^*$ is measurable and that $g : [0,1] \to \mathbb{R}^*$ is equal to f μ*-almost everywhere*. Show that g is measurable.

[10 marks]

(iv) Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x+1 & \text{if} \quad 0 \le x \le 1/2 \\ 5-x & \text{if} \quad 1/2 < x \le 1. \end{cases}$$

Show that f is measurable.

(You may assume the increasing functions and decreasing functions are measurable.)

[10 marks]

B10.

(i) What is meant by saying that a measurable function $f : [0,1] \to \mathbb{R}^*$ is an *integrable* function?

[3 marks]

(ii) State (but do not prove) Fatou's Lemma.

[3 marks]

The following result is a particular case of the Dominated Convergence Theorem.

Let $f_n : [0,1] \to \mathbb{R}$, $n \geq 1$, be a sequence of integrable functions such that $f_n(x)$ converges to $f(x)$ for all $x \in [0,1]$. Suppose there exists a non-negative integrable function $g : [0,1] \to \mathbb{R}$ such that, for all $n \geq 1$, $|f_n(x)| \leq g(x)$ for all $x \in [0,1]$. Then f is integrable and

$$\lim_{n \to +\infty} \int f_n \, d\mu = \int f \, d\mu.$$

(iii) Prove the version of the Dominated Convergence Theorem stated above.

(You may assume Fatou's Lemma but not the general form of the Dominated Convergence Theorem.)

[12 marks]

(iv) Deduce that $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(2\pi kx)}{k^{3/2}}$$

is an integrable function and that

$$\int f \, d\mu = 0.$$

(You may find it helpful to recall that the function $\sum_{k=1}^{\infty} g_k$ is defined to be the limit of the sequence of functions $f_n = \sum_{k=1}^{n} g_k$.)

[7 marks]

B11. For $f, g \in L^2([-\pi, \pi], \mu, \mathbb{R})$, write

$$\|f\|_2 = \left(\frac{1}{\pi}\int_{-\pi}^{\pi} |f|^2 \, d\mu\right)^{1/2}$$

and

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f g \, d\mu.$$

(i) Show that $d(f, g) = \|f - g\|_2$ defines a metric on $L^2([-\pi, \pi], \mu, \mathbb{R})$.
(You may assume Hölder's inequality:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |fg| \, d\mu \leq \|f\|_2 \|g\|_2.)$$

[12 marks]

In what follows we use the following notation.
Define $\varphi_0(x) = 1/\sqrt{2}$ and, for $n > 0$,

$$\varphi_n(x) = \cos(nx), \quad \varphi_{-n}(x) = \sin(nx).$$

Set

$$S_n(f, x) = \sum_{k=-n}^{n} \langle f, \varphi_n\rangle \varphi_n(x).$$

(ii) State (but do not prove) the Riesz-Fischer Theorem.

[3 marks]

(iii) Deduce that $\mathcal{F} = \{\varphi_n\}_{n=-\infty}^{\infty}$ is a basis for $L^2([-\pi, \pi], \mu, \mathbb{R})$.
(You may assume that $\mathcal{F}$ is an orthonormal set of vectors and you may assume the Riesz-Fischer Theorem.)

[10 marks]

SECTION C

Answer **BOTH** of the two questions

C12.

(i) Explain how the Lebesgue outer measure μ^* is defined.

[2 marks]

(ii) Using the definition in part (i), show that if E is a null set then $\mu^*(E) = 0$.

[4 marks]

The remainder of the question concerns the collection $\mathcal{M}_0$ of sets $E \subset \mathbb{R}$ such

$$\mu^*(E \cap A) + \mu^*(E^c \cap A) \leq \mu^*(A)$$

for every $A \subset \mathbb{R}$. You may assume that $\mathcal{M}_0$ is a σ-algebra.

(iii) Show that if E is a null set then $E \in \mathcal{M}_0$.
(You may assume that μ^* is monotone.)

[3 marks]

(iv) Show that, for every $a \in \mathbb{R}$, the interval $(a, \infty) \in \mathcal{M}_0$.

[9 marks]

(v) Deduce that, for every $a, b \in \mathbb{R}$, the interval $(a, b) \in \mathcal{M}_0$.

[4 marks]

(vi) Deduce that the Borel σ-algebra of $\mathbb{R}$ is contained in $\mathcal{M}_0$.

[3 marks]

C13. For $x, y \in [0,1)$ and $E \subset [0,1)$, define

$$x \oplus y = \begin{cases} x+y & \text{if } 0 \leq x+y < 1 \\ x+y-1 & \text{if } 1 \leq x+y < 2. \end{cases}$$

and

$$x \oplus E = \{x \oplus e : e \in E\}.$$

You may use the following results:

- for any $E \subset [0,1)$, $\mu^*(x \oplus E) = \mu^*(E)$;
- if $E \subset [0,1)$ is measurable then, for any $x \in [0,1)$, $x \oplus E$ is measurable and $\mu(x \oplus E) = \mu(E)$.

(i) Explain how to obtain a subset of $[0,1)$ which is *not* Lebesgue measurable.
(You may assume the Axiom of Choice.)

[13 marks]

(ii) Let E denote the non-measurable set obtained in part (i). Show that $\mu^*(E) > 0$.

[5 marks]

For the next part of the question, you may assume that, given $\epsilon > 0$ and $x \in \mathbb{R}$, it is possible to choose a rational number r such that $0 < x - r < \epsilon$.

(iii) Given $\epsilon > 0$, show that it is possible to choose the non-measurable set E such that $\mu^*(E) \leq \epsilon$.

[7 marks]

END OF EXAMINATION PAPER