8. The Löwenheim – Skolem Theorems.

Fix a similarity type σ.

8.1 Definition

A σ-structure $\mathfrak{A} = \langle A; \cdots \rangle$ is called <u>countable</u> if its domain A is a countable set.

8.2 Theorem (The Downward Löwenheim – Skolem Theorem.)

Let $\mathfrak{A} = \langle A; \cdots \rangle$ be any σ-structure, with σ a <u>countable</u> similarity type (see 3.7). Let S be a countable subset of A. Then there exists a σ-structure $\mathfrak{A}' = \langle A'; \cdots \rangle$ such that $S \subseteq A'$, $\mathfrak{A}' \leq \mathfrak{A}$ and $\mathfrak{A}'$ countable.

<u>Proof.</u>

Fix some element $\alpha \in A$.

For each $\mathcal{L}_\sigma$-formula ϕ and each $p \geq 1$, we define a function $\theta_{\langle \phi, p \rangle} : A^\omega \to A$ as follows:

$$\theta_{\langle \phi, p \rangle}(\bar{a}) = \begin{cases} \text{some } b \in A \text{ s.th. } \mathfrak{A} \models \phi[\bar{a}(p/b)], \text{ if such a } b \text{ exists,} \\ \alpha \quad \text{otherwise.} \end{cases}$$

For each $n \geq p$, we also define $\theta^{(n)}_{\langle \phi, p \rangle} : A^n \to A$ by

$$\theta^{(n)}_{\langle \phi, p \rangle}(a_1, \ldots, a_n) = \theta_{\langle \phi, p \rangle}(\langle a_1, \ldots, a_n, \alpha, \alpha, \alpha, \ldots \rangle).$$

Consider the type $\sigma^* = \langle I, J \sqcup J^*, K, \rho, \mu^* \rangle$ where $J^* = \{\langle p, n, \phi \rangle : n \geq p \geq 1, \phi \text{ a fmla. of } \mathcal{L}_\sigma\}$, assumed disjoint from J, and $\mu^*(\langle p, n, \phi \rangle) := n$, $\mu^*(j) = \mu(j)$ (for $j \in J$). Let $\mathfrak{A}^*$ be the σ^*-structure which is the same as $\mathfrak{A}$ as a σ-structure, and where, if $j \in J^*$, say $j = \langle p, n, \phi \rangle$, we define $f_j = \theta^{(n)}_{\langle \phi, p \rangle}$.

8.2.1 <u>Exercise</u>

Prove that σ^* is also a countable similarity type.

Thus, by 8.2.1 and 3.8, there exists a substructure $\mathfrak{A}^\dagger$ of $\mathfrak{A}^*$ (for the new type σ^*), such that $S \subseteq dom(\mathfrak{A}^\dagger)$ and $dom(\mathfrak{A}^\dagger)$ is countable.
Then we may consider $\mathfrak{A}^\dagger$ also as an $\mathcal{L}_\sigma$-structure (just by forgetting the new functions) – call it $\mathfrak{A}'$.
Certainly $\mathfrak{A}' \subseteq \mathfrak{A}$, but $dom(\mathfrak{A}')$ $(= dom(\mathfrak{A}^\dagger))$ has the additional property that it is closed under all the functions $\theta^{(n)}_{\langle\phi,p\rangle}$ (for $\langle p, n, \phi\rangle \in J^*$).

We complete the proof by showing that $\mathfrak{A}' \preceq \mathfrak{A}$ (in the language $\mathcal{L}_\sigma$). We use Tarski's Lemma. We have already remarked that $\mathfrak{A}' \subseteq \mathfrak{A}$, so to verify 7.3(2) let ϕ be any formula of $\mathcal{L}_\sigma$ and let $\bar{a} \in (dom(\mathfrak{A}'))^\omega$ and let $p \geq 1$. Suppose that for some $b \in A$ we have $\mathfrak{A} \models \phi\,[\bar{a}(p/b)]$. We must find such a b that lies in $dom(\mathfrak{A}')$.
Now choose $n \geq p$ so that all the variables occurring free in ϕ are amongst $v_1, \ldots, v_n$.
Then the sequences $\bar{a}(p/b)$ and $\langle a_1, \ldots, a_{p-1}, b, a_{p+1}, \ldots, a_n, \alpha, \alpha, \ldots\rangle$ agree at all co-ordinates q with v_q occurring free in ϕ.
Hence $\mathfrak{A} \models \phi\,[\langle a_1, \ldots, a_n, \alpha, \alpha, \ldots\rangle(p/b)]$. (by 5.6).
It follows that $\theta_{\langle\phi,p\rangle}(\langle a_1, \ldots, a_n, \alpha, \alpha, \ldots\rangle) =$ some (possibly different) $b \in A$ s.th. $\mathfrak{A} \models \phi\,[\langle a_1, a_2, \ldots, a_n, \alpha, \alpha, \ldots\rangle(p/b)]$

But $\theta_{\langle\phi,p\rangle}(\langle a_1, \ldots, a_n, \alpha, \alpha, \ldots\rangle) = \theta^{(n)}_{\langle\phi,p\rangle}(a_1, \ldots, a_n)$, and

$dom(\mathfrak{A}')$ is closed under the function $\theta^{(n)}_{\langle\phi,p\rangle}$, hence, since

$a_1, \ldots, a_n \in \mathrm{dom}(\mathfrak{A}')$, we have $\Theta^{(n)}_{\langle \phi, p \rangle}(a_1, \ldots, a_n) \in \mathrm{dom}(\mathfrak{A}')$. So if we set $d = \Theta^{(n)}_{\langle \phi, p \rangle}(a_1, \ldots, a_n)$, we have $d \in \mathrm{dom}(\mathfrak{A}')$

and $\mathfrak{A} \models \phi[\langle a_1, \ldots, a_n, \alpha, \alpha, \ldots \rangle (p/d)]$, and hence $\mathfrak{A} \models \phi[\bar{a}(p/d)]$ (by 5.6), as required to verify 7.3(2).

Thus $\mathfrak{A}' \preceq \mathfrak{A}$ and $\mathrm{dom}(\mathfrak{A}')$ is countable.

$\square$

8.3 The Upward Löwenheim-Skolem Theorem.

Let $\mathfrak{A} = \langle A; \cdots \rangle$ be any σ-structure, with A infinite and σ a countable similarity type. Then for any infinite cardinal number $\kappa \geq |A|$, there exists a σ-structure $\mathfrak{B} = \langle B; \cdots \rangle$ with $\mathfrak{A} \preceq \mathfrak{B}$ and B of cardinality κ. (In fact, we may relax the countability assumption on σ, provided $\kappa \geq \max\{|I|, |J|, |K|\}$.) ($|X|$ denotes the cardinality of the set X.)

Proof.

Later.

8.4 Remark (Optional)

The Downward Löwenheim-Skolem theorem also holds, when suitably modified, for arbitrary similarity types σ (not just countable ones). We set $|\sigma| := \max\{\aleph_0, |I|, |J|, |K|\}$. Then 8.2 holds for cardinals κ with $|A| \geq \kappa \geq |\sigma|$ and $|S| \leq \kappa$, and the conclusion is that we may take $\mathfrak{A}'$ such that $|A'| = \kappa$.