7. Elementary embeddings

We fix σ, $\mathfrak{A}$, $\mathfrak{B}$ as at the beginning of section 6.

7.1 Definition

(1) We say that an embedding $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$ is *elementary* if *all* formulas of $\mathcal{L}_\sigma$ are π-preserved from $\mathfrak{A}$ to $\mathfrak{B}$.

(2) If $\mathfrak{A} \subseteq \mathfrak{B}$ and the identity $id_A : A \to B$ is an elementary embedding, then we say that $\mathfrak{A}$ is an *elementary substructure* of $\mathfrak{B}$, written $\mathfrak{A} \preceq \mathfrak{B}$. In other words, we have that $\mathfrak{A} \preceq \mathfrak{B}$ if and only if $\mathfrak{A} \subseteq \mathfrak{B}$ and for *all* $\mathcal{L}_\sigma$-formulas ϕ, and all $\bar{a} \in A^\omega$,

$$\mathfrak{A} \models \phi[\bar{a}] \text{ implies } \mathfrak{B} \models \phi[\bar{a}].$$

7.2 Remarks.

(1) If $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$ is elementary then, in fact, all formulas of $\mathcal{L}_\sigma$ are π-preserved *between* $\mathfrak{A}$ and $\mathfrak{B}$. (Just consider $\neg\phi$.) Similarly, if $\mathfrak{A} \preceq \mathfrak{B}$ then for all $\mathcal{L}_\sigma$-formulas ϕ and all $\bar{a} \in A^\omega$, $\mathfrak{A} \models \phi[\bar{a}]$ iff $\mathfrak{B} \models \phi[\bar{a}]$.

(2) If $\pi : \mathfrak{A} \cong \mathfrak{B}$, then π is elementary (by second part of 6.3).

Notice that the definition of "$\mathfrak{A} \preceq \mathfrak{B}$" refers to truth in both $\mathfrak{A}$ and $\mathfrak{B}$. The following lemma shows that, remarkably, we can verify that $\mathfrak{A} \preceq \mathfrak{B}$ with reference to truth in $\mathfrak{B}$ alone.

7.3 Tarski's Lemma

We have that $\mathfrak{A} \preceq \mathfrak{B}$ if and only if the following two conditions hold:-

(1) $\mathfrak{A} \subseteq \mathfrak{B}$, and

(2) for all $\mathcal{L}_\sigma$-formulas ϕ and all $\bar{a} \in A^\omega$ and all

$p \geq 1$, if, for some $b \in B$, we have $\mathcal{B} \models \phi[\bar{a}(p/b)]$, then for some $d \in A$ we have $\mathcal{B} \models \phi[\bar{a}(p/d)]$.

Proof.

$\Rightarrow$: Assume $\mathcal{A} \preccurlyeq \mathcal{B}$. Then (1) holds (by 7.1(2)). For (2), suppose that ϕ, $\bar{a} \in A^\omega$, $p \geq 1$, and $b \in B$ satisfy $\mathcal{B} \models \phi[\bar{a}(p/b)]$. Then $\mathcal{B} \models \exists v_p \phi[\bar{a}]$. Since $\mathcal{A} \preccurlyeq \mathcal{B}$ we have $\mathcal{A} \models \exists v_p \phi[\bar{a}]$ (by applying 7.1(2) and 7.2(1) with "$\exists v_p \phi$" in place of "ϕ"). Hence there is some $d \in A$ such that $\mathcal{A} \models \phi[\bar{a}(p/d)]$, as required.

$\Leftarrow$: Assume (1), (2) both hold. We show, by induction on ϕ, that for any $\bar{a} \in A^\omega$ we have

(*) $\mathcal{A} \models \phi[\bar{a}]$ if and only if $\mathcal{B} \models \phi[\bar{a}]$.

If ϕ is atomic (or even QF), (*) follows from 6.3 and our assumption (1). (π is id_A here.)

If ϕ is of the form $(\psi \wedge \chi)$ or $\neg\psi$ then the inductive step follows exactly as in the proof of 6.3.

So suppose that ϕ has the form $\exists v_p \psi$, and (*) is true for ψ. Let $\bar{a} \in A^\omega$.

Then $\mathcal{A} \models \phi[\bar{a}]$ $\Rightarrow$ for some $b \in A$, $\mathcal{A} \models \psi[\bar{a}(p/b)]$,

$\Rightarrow$ for some $b \in A$, $\mathcal{B} \models \psi[\bar{a}(p/b)]$ (ind. hyp.)

$\Rightarrow$ for some $b \in B$, $\mathcal{B} \models \psi[\bar{a}(p/b)]$ (as $A \subseteq B$),

$\Rightarrow$ $\mathcal{B} \models \exists v_p \psi[\bar{a}]$ $\Rightarrow$ $\mathcal{B} \models \phi[\bar{a}]$.

Conversely,

$\mathcal{B} \models \phi[\bar{a}]$ $\Rightarrow$ for some $b \in B$, $\mathcal{B} \models \psi[\bar{a}(p/b)]$,

$\Rightarrow$ for some $d \in A$, $\mathcal{B} \models \psi[\bar{a}(p/d)]$ (by our assumption (2)),

$\Rightarrow$ for some $d \in A$, $\mathfrak{A} \models \psi[\bar{a}(p/d)]$ (by our inductive hypothesis. Note that $\bar{a}(p/d) \in A^\omega$),

$\Rightarrow$ $\mathfrak{A} \models \exists v_p \psi[\bar{a}] \Rightarrow \mathfrak{A} \models \phi[\bar{a}]$.

This completes the induction. So (*) is true for all formulas ϕ and hence $\mathfrak{A} \preccurlyeq \mathfrak{B}$.

$\square$

7.4 <u>Definition</u>

If $\pi : \mathfrak{A} \cong \mathfrak{A}$, then π is called an <u>automorphism</u> of $\mathfrak{A}$.

7.5 <u>Corollary</u> (An algebraic criterion for elementary substructurehood.)

Suppose $\mathfrak{A} \subseteq \mathfrak{B}$. Assume that for every finite subset X of A and every $b \in B \setminus A$, there exists an automorphism $\pi : \mathfrak{B} \cong \mathfrak{B}$ such that (i) for each $x \in X$, $\pi(x) = x$, and (ii) $\pi(b) \in A$. Then $\mathfrak{A} \preccurlyeq \mathfrak{B}$.

<u>Proof.</u>

We apply Tarski's Lemma. We must verify 7.3(2) (as 7.3(1) is given). So let ϕ, $\bar{a} \in A^\omega$, $p \geq 1$ and $b \in B$ be such that $\mathfrak{B} \models \phi[\bar{a}(p/b)]$. If $b \in A$ we take $d = b$. So assume $b \in B \setminus A$. Choose $n \geq p$ so that the only free variables of ϕ are amongst $v_1, \ldots, v_n$. Let $X = \{a_1, \ldots, a_n\}$ and find $\pi : \mathfrak{B} \cong \mathfrak{B}$ such that $\pi(a_i) = a_i$ for $i = 1, \ldots, n$, and $\pi(b) \in A$ (as given by the hypotheses of 7.5). Let $d = \pi(b)$, so $d \in A$. By the second part of 6.3 we have that $\mathfrak{B} \models \phi[\pi(\bar{a}(p/b))]$.

Now $\pi(\bar{a}(p/b)) = \langle \pi(a_1), \ldots, \pi(a_{p-1}), \pi(b), \pi(a_{p+1}), \ldots, \pi(a_n), \ldots \rangle$

$= \langle a_1, \ldots, a_{p-1}, d, a_{p+1}, \ldots, a_n, \pi(a_{n+1}), \ldots \rangle$

Thus $\pi(\bar{a}(p/b))$ and $\bar{a}(p/d)$ agree on their q^{th} coordinates for every q such that v_q occurs free in ϕ (by definition of n).
Hence, by 5.6, $\mathfrak{B} \models \phi[\bar{a}(p/d)]$. Since $d \in A$, this verifies 7.3(2) and hence, by 7.3, $\mathfrak{A} \preccurlyeq \mathfrak{B}$.

□

7.6 <u>Example</u>

Consider the empty language $\mathcal{L}$: $I = J = K = \emptyset$.
Let $\mathfrak{A} = \langle A;\rangle$, $\mathfrak{B} = \langle B;\rangle$ be $\mathcal{L}$-structures with $A \subseteq B$ and A (and B) infinite. Then $\mathfrak{A} \preccurlyeq \mathfrak{B}$.

<u>Proof.</u>
Obviously $\mathfrak{A} \subseteq \mathfrak{B}$. It is sufficient to verify the hypothesis (2) of 7.5. So let $X \underset{\text{finite}}{\subseteq} A$ and $b \in B \setminus A$.

Since A is infinite we may choose $d \in A \setminus X$.
Define the function $\pi : B \to B$ by

$$\pi(y) = \begin{cases} b & \text{if } y = d, \\ d & \text{if } y = b, \\ y & \text{if } y \in B \setminus \{b, d\}. \end{cases}$$

Clearly π is a bijection from B to B and hence $\pi : \mathfrak{B} \cong \mathfrak{B}$. (There's nothing else to check! – see 2.4 (b), (c), (d).)
Further, since $X \subseteq B \setminus \{b, d\}$ we have $\pi(x) = x$ for each $x \in X$, and $\pi(b) = d \in A$.

□

7.7 <u>Exercise</u>

With $\mathcal{L}$ as in 7.6, if A is finite, which sets B satisfy $\langle A;\rangle \preccurlyeq \langle B;\rangle$?

7.8 <u>Exercise</u>

Prove that $\langle \mathbb{R}; +; 0\rangle \preccurlyeq \langle \mathbb{C}; +; 0\rangle$. [Hint: view these structures as vector spaces over the field $\mathbb{Q}$.]