6. Preservation

As above, we fix a similarity type $\sigma = \langle I, J, K, \rho, \mu \rangle$ and consider two $\mathcal{L}_\sigma$-structures (i.e. σ-structures)

$$\mathfrak{A} = \langle A ; \{R_i\}_{i \in I} ; \{f_j\}_{j \in J} ; \{c_k\}_{k \in K} \rangle$$

and
$$\mathfrak{B} = \langle B ; \{S_i\}_{i \in I} ; \{g_j\}_{j \in J} ; \{d_k\}_{k \in K} \rangle.$$

6.1 Definition

Let $h : A \to B$ be any function and ϕ any $\mathcal{L}_\sigma$-formula.

(i) We say that ϕ is h-preserved from $\mathfrak{A}$ to $\mathfrak{B}$ if for all $\bar{a} \in A^\omega$, if $\mathfrak{A} \models \phi[\bar{a}]$ then $\mathfrak{B} \models \phi[h(\bar{a})]$.

(ii) We say that ϕ is h-preserved between $\mathfrak{A}$ and $\mathfrak{B}$ if for all $\bar{a} \in A^\omega$, $\mathfrak{A} \models \phi[\bar{a}]$ if and only if $\mathfrak{B} \models \phi[h(\bar{a})]$.

6.2 Definition

We say that ϕ is a quantifier-free (QF) formula of $\mathcal{L}_\sigma$ if ϕ contains no occurrences of the symbol $\exists$ (or $\forall$). I.e. ϕ can be constructed using just the rules 4.10 (1), (2) and (3).

6.3 Theorem

Suppose that $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$. Then all QF formulas of $\mathcal{L}_\sigma$ are π-preserved between $\mathfrak{A}$ and $\mathfrak{B}$. If, further, $\pi : \mathfrak{A} \cong \mathfrak{B}$, then all formulas of $\mathcal{L}_\sigma$ are π-preserved between $\mathfrak{A}$ and $\mathfrak{B}$.

Proof.

Suppose that $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$ and consider the formula $\tau_1 \doteq \tau_2$ (as in 4.9.1 (i)). Let $\bar{a} \in A^\omega$.

Then $\mathfrak{A} \models \tau_1 \doteq \tau_2 [\bar{a}]$ iff $\tau_1^{\mathfrak{A}}[\bar{a}] = \tau_2^{\mathfrak{A}}[\bar{a}]$ (by 5.1(1)) iff $\pi(\tau_1^{\mathfrak{A}}[\bar{a}]) = \pi(\tau_2^{\mathfrak{A}}[\bar{a}])$ (since, for the $\Leftarrow$ direction, π is one-one)

iff $\tau_1^{\mathfrak{B}}[\pi(\bar{a})] = \tau_2^{\mathfrak{B}}[\pi(\bar{a})]$ (by 4.7)

iff $\mathfrak{B} \models \tau_1 \doteq \tau_2 \, [\pi(\bar{a})]$ (by 5.1(1)).

So the theorem holds for formulas of this form.

Now consider the formula $P_i(\tau_1, \ldots, \tau_{\rho(i)})$ (as in 4.9.1 (ii))

Then $\mathfrak{A} \models P_i(\tau_1, \ldots, \tau_{\rho(i)})[\bar{a}]$ iff $\langle \tau_1^{\mathfrak{A}}[\bar{a}], \ldots, \tau_{\rho(i)}^{\mathfrak{A}}[\bar{a}] \rangle \in R_i$ (5.1(2))

iff $\langle \pi(\tau_1^{\mathfrak{A}}[\bar{a}]), \ldots, \pi(\tau_{\rho(i)}^{\mathfrak{A}}[\bar{a}]) \rangle \in S_i$ (by 2.7, 2.4(b))

iff $\langle \tau_1^{\mathfrak{B}}[\pi(\bar{a})], \ldots, \tau_{\rho(i)}^{\mathfrak{B}}[\pi(\bar{a})] \rangle \in S_i$ (by 4.7)

iff $\mathfrak{B} \models P_i(\tau_1, \ldots, \tau_{\rho(i)})[\pi(\bar{a})]$ (by 5.1 (2)).

So the theorem holds for all atomic formulas.

We now proceed by induction. So suppose the theorem holds for the formulas ψ, χ.

We have $\mathfrak{A} \models (\psi \wedge \chi)[\bar{a}]$ iff $\mathfrak{A} \models \psi[\bar{a}]$ and $\mathfrak{A} \models \chi[\bar{a}]$ (5.1(3))

iff $\mathfrak{B} \models \psi[\pi(\bar{a})]$ and $\mathfrak{B} \models \chi[\pi(\bar{a})]$ (inductive hypothesis)

iff $\mathfrak{B} \models (\psi \wedge \chi)[\pi(\bar{a})]$ (5.1(3)).

So the theorem holds for the formula $(\psi \wedge \chi)$.

Also, $\mathfrak{A} \models \neg\psi[\bar{a}]$ iff not $\mathfrak{A} \models \psi[\bar{a}]$ (by 5.1(4))

iff not $\mathfrak{B} \models \psi[\pi(\bar{a})]$ (inductive hyp.)

iff $\mathfrak{B} \models \neg\psi[\pi(\bar{a})]$ (by 5.1(4)).

Thus the theorem holds for the formula $\neg\psi$. This completes the proof (by induction) of the first part of the theorem.

Now suppose the theorem holds for some formula ψ. Consider the formula $\exists v_p \psi$.

Then $\mathfrak{A} \models \exists v_p \psi[\bar{a}]$ iff for some $b \in A$, $\mathfrak{A} \models \psi[\bar{a}(p/b)]$ (by 5.1(5)),

iff for some $b \in A$, $\mathfrak{B} \models \psi[\pi(\bar{a}(p/b))]$ (by ind. hyp.)

iff for some $b \in A$, $\mathfrak{B} \models \psi[\pi(\bar{a})(p/\pi(b))]$

which implies that for some $d \in B$ (namely $d = \pi(b)$) we have $\mathfrak{B} \models \psi[\pi(\bar{a})(p/d)]$, and so $\mathfrak{B} \models \exists v_p \psi[\pi(\bar{a})]$.

This establishes the "$\Rightarrow$" direction of the required conclusion, and it is as much as we can say under the assumption that $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$. If we now assume that $\pi : \mathfrak{A} \cong \mathfrak{B}$ (so $\pi : A \to B$ is surjective) then we can complete the proof of the second part of the theorem as follows.

Suppose $\mathfrak{B} \models \exists v_p \psi[\pi(\bar{a})]$. Then for some $d \in B$, we ~~have~~ $\mathfrak{B} \models \psi[\pi(\bar{a})(p/d)]$. Since $\pi : A \to B$ is onto, there is some $b \in A$ such that $\pi(b) = d$. So $\mathfrak{B} \models \psi[\pi(\bar{a})(p/\pi(b))]$, i.e. $\mathfrak{B} \models \psi[\pi(\bar{a}(p/b))]$.

But now, by our inductive hypothesis, $\mathfrak{A} \models \psi[\bar{a}(p/b)]$ for some $b \in A$. Hence, by 5.1(5), $\mathfrak{A} \models \exists v_p \psi[\bar{a}]$.
Thus we have shown that if the theorem holds for ψ and if $\pi : \mathfrak{A} \cong \mathfrak{B}$, then we have $\mathfrak{A} \models \exists v_p \psi[\bar{a}]$ iff $\mathfrak{B} \models \exists v_p \psi[\pi(\bar{a})]$ (for all $\bar{a} \in A^\omega$), i.e. the theorem holds for the formula $\exists v_p \psi$.

This completes the proof (by induction) that all formulas of $\mathcal{L}_\sigma$ are π-preserved between $\mathfrak{A}$ and $\mathfrak{B}$ if $\pi : \mathfrak{A} \cong \mathfrak{B}$.

$\square$

6.4 Exercise

Deduce from the first part of 6.3 that if $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$ and ϕ is a formula of the form

$$\exists v_{i_1} \exists v_{i_2} \cdots \exists v_{i_r} \psi \qquad (*)$$

where ψ is a QF formula, then ϕ is π-preserved from $\mathfrak{A}$ to $\mathfrak{B}$. (Such formulas are called existential.)

6.5 Exercise

Let $\mathfrak{A} = \langle \mathbb{Z} ; + ; 0 \rangle$. Prove that there is an existential formula E (of $\mathcal{L}_\sigma$ where σ is the type of the structure $\mathfrak{A}$) such that for all $\bar{a} \in \mathbb{Z}^\omega$, $\mathfrak{A} \models E[\bar{a}]$ iff a_1 is even. Show, however, that there is no such existential formula if we replace "even" by "odd". (Though, of course, there is a formula that works, namely $\neg E$.)
[Hint: Use 6.4 for a suitable $\pi : \mathfrak{A} \hookrightarrow \mathfrak{A}$.]