is an $\mathcal{L}_\sigma$-formula (in which all the rules have been used). The only variables occurring free are v_4, v_3, v_1.

5. The intepretation of $\mathcal{L}_\sigma$-formulas.

As usual, let $\mathfrak{A} = \langle A; \{R_i\}_{i\in I}; \{f_j\}_{j\in J}; \{c_k\}_{k\in K}\rangle$ be a σ structure. Let $\bar{a} \in A^\omega$ and let ϕ be a formula of $\mathcal{L}_\sigma$. We write "$\mathfrak{A} \models \phi[\bar{a}]$" to mean that when ϕ is interpreted in $\mathfrak{A}$ in the obvious way, and when a_p is substituted for the variable v_p for each $p \geq 1$ (where $\bar{a} = \langle a_1, a_2, \ldots, a_p, \ldots\rangle$), then we get a true statement (see 1.8 and 1.9). Of course, this has to be made rigorous, which we do now by induction on ϕ.

5.1 <u>Definition</u> (Tarski's truth definition.)
With the above notation we write $\mathfrak{A} \models \phi[\bar{a}]$ ("ϕ is true in the σ-structure $\mathfrak{A}$ at the assignment $\bar{a}$") if

(1) ϕ is an atomic formula of the form $\tau_1 \doteq \tau_2$ and then $\mathfrak{A} \models \phi[\bar{a}] \;:\Longleftrightarrow:\; \tau_1^{\mathfrak{A}}[\bar{a}] = \tau_2^{\mathfrak{A}}[\bar{a}]$;

or (2) ϕ is an atomic formula of the form $R_i(\tau_1, \ldots, \tau_{\mu(i)})$ (for some $i \in I$) and then $\mathfrak{A} \models \phi[\bar{a}] \;:\Longleftrightarrow:\; \langle \tau_1^{\mathfrak{A}}[\bar{a}], \ldots, \tau_{\mu(i)}^{\mathfrak{A}}[\bar{a}]\rangle \in R_i$;

or (3) ϕ is $(\psi \wedge \chi)$ (as in 4.10 (2)) and then $\mathfrak{A} \models \phi[\bar{a}] \;:\Longleftrightarrow:\; \mathfrak{A} \models \psi[\bar{a}]$ and $\mathfrak{A} \models \chi[\bar{a}]$;

or (4) ϕ is $\neg\psi$ (as in 4.10 (3)) and then $\mathfrak{A} \models \phi[\bar{a}]$ $:\Longleftrightarrow:$ it is not the case that $\mathfrak{A} \models \psi[\bar{a}]$;

or (5) ϕ is $\exists v_p \psi$ (as in 4.10(4)) and then $\mathfrak{A} \models \phi[\bar{a}]$ $:\Longleftrightarrow:$ there exists $b \in A$ such that $\mathfrak{A} \models \psi[\bar{a}(p/b)]$, where $\bar{a}(p/b)$ denotes the sequence $\langle a_1, \ldots, a_{p-1}, b, a_{p+1}, \ldots\rangle$.

5.2 **Abbreviations**

Let ψ, χ be $\mathcal{L}_\sigma$-formulas. Then so are $\neg(\neg\psi \wedge \neg\chi)$, $\neg(\psi \wedge \neg\chi)$, $(\neg(\psi \wedge \neg\chi) \wedge \neg(\chi \wedge \neg\psi))$ and, for any $p \geq 1$, $\neg\exists v_p \neg\psi$. These are abbreviated to $(\psi \vee \chi)$, $(\psi \to \chi)$, $(\psi \leftrightarrow \chi)$ and $\forall v_p \psi$ respectively.

5.3 **Lemma**

Let $\mathfrak{A}$ be a σ-structure and let $\bar{a} \in A^\omega$. Then $\mathfrak{A} \models (\psi \vee \chi)[\bar{a}] \iff$ either $\mathfrak{A} \models \psi[\bar{a}]$ or $\mathfrak{A} \models \chi[\bar{a}]$; $\mathfrak{A} \models (\psi \to \chi)[\bar{a}]$ $\iff$ if $\mathfrak{A} \models \psi[\bar{a}]$ then $\mathfrak{A} \models \chi[\bar{a}]$; $\mathfrak{A} \models (\psi \leftrightarrow \chi)[\bar{a}] \iff \mathfrak{A} \models \psi[\bar{a}]$ if and only if $\mathfrak{A} \models \chi[\bar{a}]$; and $\mathfrak{A} \models \forall v_p \psi[\bar{a}] \iff$ for all $b \in \mathrm{dom}(\mathfrak{A})$, $\mathfrak{A} \models \psi[\bar{a}(p/b)]$.

Proof.

The first three equivalences are very easy. I leave the fourth as an exercise.

5.4 **Examples.**

Let $\mathcal{L}_\sigma$ be specified as containing just a binary function symbol F_1. Let $\mathfrak{A}$ be $\langle \mathbb{N}_{>0} ; \cdot \rangle$. Let ϕ be the formula

$$\forall v_2 \overbrace{\forall v_3 \overbrace{(F_1(v_2, v_3) \doteq v_1 \to (v_2 \doteq v_1 \vee v_3 \doteq v_1))}^{\psi'}}^{\psi}$$

Let $\bar{a} \in (\mathbb{N}_{>0})^\omega$.

Then $\mathfrak{A} \models \phi[\bar{a}] \iff$ for all $b \in \mathbb{N}_{>0}$ $\mathfrak{A} \models \psi[\bar{a}(2/b)]$ (by 5.3)

$\iff$ for all $b \in \mathbb{N}_{>0}$, for all $d \in \mathbb{N}_{>0}$, $\mathfrak{A} \models \psi'[\bar{a}(2/b)(3/d)]$ (by 5.3)

$\iff$ — " —, — " — if $\mathfrak{A} \models F_1(v_2, v_3) \doteq v_1 [\bar{a}(2/b)(3/d)]$ then

$\mathcal{G} \models (v_2 \simeq v_1 \vee v_3 \simeq v_1)[\bar{a}(2/b)(3/d)]$.

So $\mathcal{G} \models \phi[\bar{a}] \iff$ for all $b, d \in \mathbb{N}_{>0}$, if $b \cdot d = a_1$ then $b = a_1$ or $d = a_1$.
$\iff$ $a_1 = 1$ or a_1 is a prime number.

Let $Prime(v_1)$ denote the formula $(\phi \wedge \neg F_1(v_1, v_1) \simeq v_1)$ (with ϕ as above) so that $\mathcal{G} \models Prime(v_1)[\bar{a}]$ if and only if a_1 is a prime number.
Now let $Prime(v_4)$ be the result of replacing every occurrence of v_1 in $Prime(v_1)$ by the variable v_4.
Then the same argument as above shows that $\mathcal{G} \models Prime(v_4)[\bar{a}]$ if and only if a_4 is a prime number.
(Exercise 5.4(1): What happens if we use v_2 or v_3 instead of v_4 here?)

Now add to the language above another binary function symbol F_2 and a constant symbol c_1.
Let $\mathcal{G}' = \langle \mathbb{N}_{>0} ; \cdot, + ; 2 \rangle$.
Consider the formula ψ:

$$\forall v_2 \underbrace{\exists v_1}_{\chi_0} \exists v_4 \underbrace{(F_2(F_1(c_1, v_2), c_1) = F_2(v_1, v_4)}_{\chi_1} \wedge (Prime(v_1) \wedge Prime(v_4)))$$

Then $\mathcal{G}' \models \psi[\bar{a}] \iff$ for all $b \in \mathbb{N}_{>0}$ $\mathcal{G} \models \chi_0[\bar{a}(2/b)]$
$\iff$ (applying 5.1(5) twice), for all $b \in \mathbb{N}_{>0}$, there exist $d, e \in \mathbb{N}_{>0}$, $\mathcal{G} \models \chi_1[\bar{a}(2/b)(1/d)(4/e)]$
$\iff$ for all $b \in \mathbb{N}_{>0}$, there exist $d, e \in \mathbb{N}_{>0}$ such that $\mathcal{G} \models \chi_1[\langle d, b, a_3, e, a_4, a_5, \ldots \rangle]$

$\iff$ for all $b \in \mathbb{N}_{>0}$, there exist $d, e \in \mathbb{N}_{>0}$ such that $2 \cdot b + 2 = d + e$ and d is prime and e is prime.
$\iff$ every even number greater than 2 is the sum of two prime numbers.

$\iff$ Goldbach's conjecture is true.

Tarski's truth definition is often rendered as: "It is snowing" is true if and only if it is snowing. (Even if we are sitting in a room with no windows.)

5.5 <u>Exercise</u>

Let $\mathfrak{A}$ be the structure $\langle \mathbb{Z}; +, \cdot \rangle$. What is the type σ of $\mathfrak{A}$? Write down a formula in this language – ψ say – having the property that for all $\bar{a} \in A^{\omega}$

$$\mathfrak{A} \models \psi[\bar{a}] \iff a_1 \geq 0.$$

[Email me for a hint.]

5.6 <u>Exercise</u>

Let σ be any similarity type and $\mathfrak{A}$ a σ-structure. Let ϕ be an L_σ-formula and $\bar{a} \in (\mathrm{dom}(\mathfrak{A}))^{\omega}$. Prove (after making precise) that whether or not $\mathfrak{A} \models \phi[\bar{a}]$ depends only on those a_p such that v_p occurs free in ϕ.