4.7 <u>Lemma</u>

Let $\mathfrak{A}, \mathfrak{B} \in K_\sigma$ and let $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$. Let $\tau$ be any $\mathcal{L}_\sigma$-term. Then for all $\bar{a} \in A^\omega$,

$$\pi(\tau^{\mathfrak{A}}[\bar{a}]) = \tau^{\mathfrak{B}}[\pi(\bar{a})].$$

<u>Proof.</u>

By induction on $\tau$. We use the notation of 2.4 for $\mathfrak{A}, \mathfrak{B}$. The base cases are when $\tau$ is $v_p$ or $c_k$.

Now
$$\begin{aligned} \pi(v_p^{\mathfrak{A}}[\bar{a}]) &= \pi(a_p) && (\text{by } 4.4(i)) \\ &= \pi(\bar{a})_p \\ &= v_p^{\mathfrak{B}}[\pi(\bar{a})] && (\text{by } 4.4(i)), \end{aligned}$$

and
$$\begin{aligned} \pi(c_k^{\mathfrak{A}}[\bar{a}]) &= \pi(e_k) && (\text{by } 4.4(ii)) \\ &= d_k && (\text{by } 2.4(d)) \\ &= c_k^{\mathfrak{B}}[\pi(\bar{a})] && (\text{by } 4.4(ii)). \end{aligned}$$

For the inductive step, suppose that $\tau$ is $F_j(\tau_1, .., \tau_{\mu(j)})$ (for some $j \in J$) and that the lemma holds for $\tau_1, .., \tau_{\mu(j)}$. I.e. assume that

$(*)$ ... $\pi(\tau_r^{\mathfrak{A}}[\bar{a}]) = \tau_r^{\mathfrak{B}}[\pi(\bar{a})]$ for $r = 1, \ldots, \mu(j)$.

Then
$$\begin{aligned} \pi(\tau^{\mathfrak{A}}[\bar{a}]) &= \pi\left(F_j(\tau_1, .., \tau_{\mu(j)})^{\mathfrak{A}}[\bar{a}]\right) \\ &= \pi\left(f_j(\tau_1^{\mathfrak{A}}[\bar{a}], \ldots, \tau_{\mu(j)}^{\mathfrak{A}}[\bar{a}])\right) && (\text{by } 4.4(iii)) \\ &= g_j\left(\pi(\tau_1^{\mathfrak{A}}[\bar{a}]), \ldots, \pi(\tau_{\mu(j)}^{\mathfrak{A}}[\bar{a}])\right) && (\text{by } 2.4(c)). \\ &= g_j\left(\tau_1^{\mathfrak{B}}[\pi(\bar{a})], \ldots, \tau_{\mu(j)}^{\mathfrak{B}}[\pi(\bar{a})]\right) && (\text{by } (*)) \\ &= F_j(\tau_1, .., \tau_{\mu(j)})^{\mathfrak{B}}[\pi(\bar{a})] && (\text{by } 4.4(iii)) \\ &= \tau^{\mathfrak{B}}[\pi(\bar{a})], && \text{as required.} \end{aligned}$$

$\square$

4.8 <u>Exercises</u>

Let $\mathcal{T}_1$ be the collection of $\mathcal{L}_\sigma$-terms in which at most the variable $v_1$ occurs. Then (by 4.5) for $\tau \in \mathcal{T}_1$, $\tau^{\mathfrak{A}}$ may be considered as a function from A to A (for any σ-structure $\mathfrak{A}$ with domain A).

Describe in a mathematical way the collection of functions $\{\tau^{\mathfrak{A}} : \tau \in \mathcal{T}_1\}$ in the following cases:

(1) $\mathfrak{A} = \langle \mathbb{R}; +, \cdot, -; 0, 1 \rangle$.

(2) $\mathfrak{A} = \langle \mathbb{Z}; \cdot, f \rangle$ where $f(z) := 2z$ (for $z \in \mathbb{Z}$).

4.9 <u>Atomic formulas of $\mathcal{L}_\sigma$</u>

4.9.1 An <u>atomic formula of $\mathcal{L}_\sigma$</u> is, by definition, an $\mathcal{L}_\sigma$-word of one of the following forms:

either (i) $\tau_1 \doteq \tau_2$ where $\tau_1, \tau_2$ are $\mathcal{L}_\sigma$-terms,

or (ii) $P_i(\tau_1, \ldots, \tau_{\rho(i)})$, where $i \in I$ and $\tau_1, \ldots, \tau_{\rho(i)}$ are $\mathcal{L}_\sigma$-terms.

4.9.2 <u>Remarks</u>

(i) We have Unique Readability for atomic formulas.

(ii) We say the variable $v_p$ <u>occurs</u> in an atomic formula of type (i), if either it occurs in $\tau_1$ or in $\tau_2$. Similarly, it occurs in an atomic formula of type (ii) if for some $r$, $1 \leq r \leq \rho(i)$, it occurs in $\tau_r$.

4.10 <u>Formulas of $\mathcal{L}_\sigma$</u>

We are now ready to define the collection of all formulas of $\mathcal{L}_\sigma$. The definition is inductive:

An $\mathcal{L}_\sigma$-word is called an $\mathcal{L}_\sigma$-formula (or formula of $\mathcal{L}_\sigma$) if it can be obtained by finitely many applications of the following rules:-

(1) Every atomic formula of $\mathcal{L}_\sigma$ is an $\mathcal{L}_\sigma$-formula;
(2) if $\psi, \chi$ are $\mathcal{L}_\sigma$-formulas, so is $(\psi \wedge \chi)$;
(3) if $\psi$ is an $\mathcal{L}_\sigma$-formula, so is $\neg\psi$;
(4) if $\psi$ is an $\mathcal{L}_\sigma$-formula and $p \geq 1$, then $\exists v_p \psi$ is an $\mathcal{L}_\sigma$-formula.
(5) Nothing else is an $\mathcal{L}_\sigma$-formula.

4.10.1 Remarks

(i) We have Unique Readability for $\mathcal{L}_\sigma$-formulas. Hence we may define notions by induction on (the length of) formulas. E.g.:

(ii) We say that a variable $v_q$ occurs free in an $\mathcal{L}_\sigma$ formula $\phi$ if

either (a) $\phi$ is atomic and $v_q$ occurs in $\phi$ (see 4.9.2 (ii)),

or (b) $\phi$ has the form $(\psi \wedge \chi)$ and either $v_q$ occurs free in $\psi$ or in $\chi$,

or (c) $\phi$ has the form $\neg\psi$ and $v_q$ occurs free in $\psi$;

or (d) $\phi$ has the form $\exists v_p \psi$ and $v_q$ occurs free in $\psi$ and $q \neq p$.

4.10.2 Example

With $\rho(1) = \rho(2) = 2$, $\mu(1) = 1$, $\mu(2) = 2$, $K = \{1\}$:

$$\neg \exists v_2 \left( \left( F_1(v_4) \doteq v_3 \wedge P_1(c_1, F_2(v_1, v_2)) \right) \wedge \exists v_4 P_2(v_4, v_1) \right)$$