4. The first-order language associated to a similarity type.

4.1 Logical symbols

All languages contain the following logical symbols:

- A countably infinite set $v_1, v_2, v_3, \ldots$ of variables;
- The equality symbol $\doteq$;
- The connectives $\wedge$ ("and"), $\neg$ ("not");
- The existential quantifier $\exists$ ("there exists");
- Parentheses ( ) and a comma , .

We now fix a similarity type $\sigma = \langle I, J, K, \rho, \mu \rangle$.

4.2 Nonlogical symbols

We specify the language $\mathcal{L}_\sigma$ associated to σ. As well as the logical symbols, it contains

- A $\rho(i)$-ary predicate symbol P_i for each $i \in I$;
- A $\mu(j)$-ary function symbol F_j for each $j \in J$;
- A constant symbol c_k for each $k \in K$.

Now, by an $\mathcal{L}_\sigma$-word we simply mean a sequence of (logical and nonlogical) symbols of $\mathcal{L}_\sigma$, e.g. $v_3 \exists (\doteq$ or $F_j(v_1) \doteq c_k$ (provided $j \in J$, $k \in K$).

We must go through the tedious process of making precise when an $\mathcal{L}_\sigma$-word "makes grammatical sense". (The second example above does, the first doesn't.)

4.3 Terms

An $\mathcal{L}_\sigma$-term is an $\mathcal{L}_\sigma$-word that can be constructed by finitely many applications of the following rules:

4.3.1 for any $p \geq 1$, v_p is an $\mathcal{L}_\sigma$-term;
4.3.2 for any $k \in K$, c_k is an $\mathcal{L}_\sigma$-term;
4.3.3 for and $j \in J$, if $\tau_1, \ldots, \tau_{\mu(j)}$ are $\mathcal{L}_\sigma$-terms, then so is the $\mathcal{L}_\sigma$-word $F_j(\tau_1, \ldots, \tau_{\mu(j)})$.
4.3.4 Nothing else is an $\mathcal{L}_\sigma$-term.

For example, say $J = \{1, 2\}$, $\mu(1) = 1$, $\mu(2) = 2$, and $K = \{1\}$. Then we may build up terms:

(by 4.3.1 with $p=6$) v_6 c_1 (by 4.3.2 with $k=1$)

(by 4.3.3 with $j=2$) $F_2(v_6, c_1)$

v_3 (by 4.3.1 with $p=3$)

(by 4.3.3 with $j=1$) $F_1(F_2(v_6, c_1))$

(by 4.3.3 with $j=2$) $F_2(v_3, F_1(F_2(v_6, c_1)))$

<u>The Unique Readability Theorem</u>

Most definitions to do with logical syntax rely on the fact that such constructions can be done in only one way. For example, suppose we wish, as we do, to define the notion "v_q occurs in the $\mathcal{L}_\sigma$-term τ". This is achieved by induction on the (number of basic symbols in) τ as follows:

- v_q occurs in v_p iff $q = p$,
- v_q does not occur in c_k,
- v_q occurs in $F_j(\tau_1, \ldots, \tau_{\mu(j)})$ iff for some r with $1 \leq r \leq \mu(j)$ we have that v_q occurs in τ_r.

This definition relies on the fact that a term can be broken down into its constituent parts in only one way. For otherwise the notions "v_q occurs in τ" might depend on how τ was built up, and not just on τ itself, as ought to be the case. However, all is well :-

4.3.5 <u>Proposition</u> (Unique readability for terms.)

Let τ be an $\mathcal{L}_\sigma$-term. Then exactly one of the following holds:

(i) For some $p \geq 1$, τ is v_p;

(ii) For some $k \in K$, τ is c_k;

(iii) There exists a unique $j \in J$, and a unique sequence $\tau_1, \ldots, \tau_{\mu(j)}$ of $\mathcal{L}_\sigma$-terms such that $\tau = F_j(\tau_1, \ldots, \tau_{\mu(j)})$.

<u>Proof</u>

Omitted.

$\square$

4.4 <u>The interpretation of terms.</u>

Let $\mathfrak{A} = \langle A ; \{R_i\}_{i \in I} ; \{f_j\}_{j \in J} ; \{e_k\}_{k \in K} \rangle$ be a σ-structure and let τ be an $\mathcal{L}_\sigma$-term. We wish to associate an actual function $\tau^{\mathfrak{A}}$ to τ. The function $\tau^{\mathfrak{A}}$ will take values in A and it is convenient, for notational purposes, to take its domain as

$$A^\omega := \{\bar{a} : \bar{a} = \langle a_1, a_2, \ldots, a_n, \ldots \rangle, \quad a_p \in A \text{ for } p \geq 1\}$$

- i.e. the set of infinite sequences of elements of A.

Then for $\bar{a} \in A^\omega$ we define the element $\tau^{\mathfrak{A}}[\bar{a}]$ of A by induction on τ as follows:

(i) if τ is v_p, then $v_p^{\mathfrak{A}}[\bar{a}] := a_p$;
(ii) if τ is c_k, then $c_k^{\mathfrak{A}}[\bar{a}] := e_k$;
(iii) if τ is $F_j(\tau_1, \ldots, \tau_{\mu(j)})$ and $\tau_1^{\mathfrak{A}}[\bar{a}], \ldots, \tau_{\mu(j)}^{\mathfrak{A}}[\bar{a}]$ have already been defined, then $\tau^{\mathfrak{A}}[\bar{a}] := f_j\left(\tau_1^{\mathfrak{A}}[\bar{a}], \ldots, \tau_{\mu(j)}^{\mathfrak{A}}[\bar{a}]\right)$.

That this definition is sound - i.e. that $\tau^{\mathfrak{A}}[\bar{a}]$ depends only on τ, $\mathfrak{A}$ and $\bar{a}$ - follows from 4.3.5.

4.4.1 <u>Example</u>

Consider the example $\tau = F_2(v_3, F_1(F_2(v_6, c_1)))$ constructed above. Let $\mathfrak{A}$ be $\langle \mathbb{R}; -, +; 0\rangle$. Let $\bar{r} \in \mathbb{R}^\omega$.

Then $\tau^{\mathfrak{A}}[\bar{r}] = +(v_3^{\mathfrak{A}}[\bar{r}], -(+(v_6^{\mathfrak{A}}[\bar{r}], c_1^{\mathfrak{A}}[\bar{r}])))$

$$= r_3 + (-(r_6 + 0))$$

$$= r_3 - r_6 .$$

4.5 <u>Lemma</u>

With the notation as above, $\tau^{\mathfrak{A}}[\bar{a}]$ depends only on the a_p such that v_p occurs in τ. In other words if $\bar{b}$ is another element of A^ω (say $\bar{b} = \langle b_1, b_2, \ldots\rangle$) and $a_p = b_p$ for all those p such that v_p occurs in τ, then $\tau^{\mathfrak{A}}[\bar{a}] = \tau^{\mathfrak{A}}[\bar{b}]$.

<u>Proof</u>.

Exercise. (Use induction on τ.)

$\square$

4.6 <u>Notation</u>

Let A, B be any sets and $\pi : A \to B$ any function. If $\bar{a} = \langle a_1, a_2, \ldots\rangle$ is a finite or infinite sequence of elements of A, we write $\pi(\bar{a})$ for $\langle \pi(a_1), \pi(a_2), \ldots\rangle$ (which is a sequence from B of the same length as $\bar{a}$).