3. A little universal algebra.

Let us elucidate the notion of substructure.
Fix a similarity type $\sigma = \langle I, J, K, \rho, \mu \rangle$ as before.
Let $\mathfrak{A}, \mathfrak{B} \in K_\sigma$ be as in 2.4, and assume $A \subseteq B$.
Then, according to 2.4 with $\pi = id_A$ we have that $\mathfrak{A} \subseteq \mathfrak{B}$ iff

3.1 (1) for each $i \in I$ and all $a_1, \ldots, a_n \in A$ (where $n = \rho(i)$),
$\langle a_1, \ldots, a_n \rangle \in R_i$ iff $\langle a_1, \ldots, a_n \rangle \in S_i$.
I.e. $R_i = A^n \cap S_i$ ----- $(*)$

(2) for each $j \in J$ and all $a_1, \ldots, a_m \in A$ (where $m = \mu(j)$),
$f_j(a_1, \ldots, a_m) = g_j(a_1, \ldots, a_m)$.
I.e. f_j is the restriction of g_j to A^n : $f_j = g_j \restriction A^m$ ...$(**)$
(In particular, if $a_1, \ldots, a_m \in A$, then $g_j(a_1, \ldots, a_m) \in A$.)

(3) for each $k \in K$, $e_k = d_k$ ...$(***)$ (In particular, $d_k \in A$.)

3.2 Now suppose that $\mathfrak{B} \in K_\sigma$ (as above) and that A is a subset of B. We ask the question: what conditions on A ensure that A is the domain of a substructure of $\mathfrak{B}$? By the comments above we must have that whenever $a_1, \ldots, a_m \in A$ then $g_j(a_1, \ldots, a_m) \in A$ (for each $j \in J$, where $m = \mu(j)$) – i.e. A is closed under g_j – and for each $k \in K$, $d_k \in A$.

3.3 Theorem

With the notation above, A is the domain of a substructure of $\mathfrak{B}$ if and only if A is closed under each g_j (for $j \in J$) and contains each d_k (for $k \in K$).

Proof.

The "only if" is proved in 3.2. For the "if" direction, suppose that A satisfies the stated condition. Then we simply define R_i by $(*)$ (for each $i \in I$), f_j by $(**)$ (for each $j \in J$) and e_k by $(***)$ (for each $k \in K$), thus defining the substructure $\langle A; \{R_i\}_{i \in I}; \{f_j\}_{j \in J}; \{e_k\}_{k \in K} \rangle$ of $\mathfrak{B}$.

$\square$

3.4 *Remark*

(a) It follows from 3.3 that if σ is a *purely relational* type, i.e. $J = K = \emptyset$, then *every* subset of the domain of a σ-structure is the domain of a substructure of that structure.

(b) It of course follows from 3.1 (1), (2) and (3) that a subset of the domain of a σ-structure can be the domain of at most one substructure.

3.5 *Theorem*

Let $\mathfrak{B} \in K_\sigma$ (in above notation) and let S be any non-empty subset of B. Then there exists a *smallest* substructure $\mathfrak{C} \leq \mathfrak{B}$ with $S \subseteq \mathrm{dom}(\mathfrak{C})$. (This means that if $\mathfrak{C}' \leq \mathfrak{B}$ and $S \subseteq \mathrm{dom}(\mathfrak{C}')$, then $\mathfrak{C} \leq \mathfrak{C}'$.) Further, $\mathfrak{C}$ is unique with this property.

Proof.

Let $S_0 := S \cup \{d_k : k \in K\}$ and, for each $n \geq 0$, let $S_{n+1} := S_n \cup \{g_j(a_1, \ldots, a_{\mu(j)}) : j \in J,\ a_1, \ldots, a_{\mu(j)} \in S_n\}$.

Let $S_\omega := \bigcup_{n=0}^{\infty} S_n$.

Clearly $d_k \in S_\omega$ for each $k \in K$.

Also, $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n \subseteq S_{n+1} \subseteq \cdots \subseteq S_\omega$. $(\dagger)$.

Further, if $j \in J$ then S_ω is closed under g_j. For if $m = \mu(j)$ and $a_1, \ldots, a_m \in S_\omega$, then by (†), there exists n such that $a_1, \ldots, a_m \in S_n$. So $g_j(a_1, \ldots, a_m) \in S_{n+1}$ (by definition of S_{n+1}). Since $S_{n+1} \subseteq S_\omega$ (by (†)) we have $g_j(a_1, \ldots, a_n) \in S_\omega$, as required.

Thus, by 3.3, S_ω is the domain of a substructure, $\mathfrak{A}$ say, of $\mathfrak{B}$. Since $S \subseteq S_0 \subseteq S_\omega$ we have $S \subseteq \mathrm{dom}(\mathfrak{A})$.

Now suppose that $\mathfrak{A}'$ is any substructure of $\mathfrak{B}$ with $S \subseteq \mathrm{dom}(\mathfrak{A}')$. By 3.1(3) we also have $d_k \in \mathrm{dom}(\mathfrak{A}')$ (for each $k \in K$), so $S_0 \subseteq \mathrm{dom}(\mathfrak{A}')$. We now prove by induction that for all n, $S_n \subseteq \mathrm{dom}(\mathfrak{A}')$.
So suppose (for the inductive step) that $S_n \subseteq \mathrm{dom}(\mathfrak{A}')$. Since $\mathrm{dom}(\mathfrak{A}')$ is closed under g_j (for each $j \in J$) by 3.1(2), it follows that $g_j(a_1, \ldots, a_{\mu(j)}) \in \mathrm{dom}(\mathfrak{A}')$ for each $a_1, \ldots, a_{\mu(j)} \in S_n$ (and each $j \in J$). Thus $S_{n+1} \subseteq \mathrm{dom}(\mathfrak{A}')$ as required.
Thus $\mathrm{dom}(\mathfrak{A}) = S_\omega = \bigcup_{n=0}^{\infty} S_n \subseteq \mathrm{dom}(\mathfrak{A}')$. The required result now follows from 3.6 below.

□

3.6 __Exercise__
Let $\mathfrak{A}, \mathfrak{A}', \mathfrak{B} \in K_\sigma$. Suppose that $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A}' \subseteq \mathfrak{B}$ and $\mathrm{dom}(\mathfrak{A}) \subseteq \mathrm{dom}(\mathfrak{A}')$. Then $\mathfrak{A} \subseteq \mathfrak{A}'$.

3.7 __Definition__
A similarity type $\sigma = \langle I, J, K, \rho, \mu \rangle$ is called __countable__ if I, J, K are each countable sets.

3.8 __Exercise__ (Downward Löwenheim-Skolem Theorem: weak form.)
By inspecting the proof of 3.5, prove that if

σ is countable and S is countable, then $\mathrm{dom}(\mathfrak{A})$ is countable.

3.9 Exercise

Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K_\sigma$ and suppose that $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$ and $\gamma : \mathfrak{B} \hookrightarrow \mathfrak{C}$. Prove that $\gamma \circ \pi : \mathfrak{A} \hookrightarrow \mathfrak{C}$.

3.10 Exercise

Prove that any embedding can be decomposed into an isomorphism and identity embedding. I.e. show that if $\mathfrak{A}, \mathfrak{B} \in K_\sigma$ and $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$, then there exists $\mathfrak{A}^* \in K_\sigma$ such that $\mathfrak{A}^* \subseteq \mathfrak{B}$ and $\pi : \mathfrak{A} \cong \mathfrak{A}^*$.