3. A little universal algebra.

Let us elucidate the notion of substructure.

Fix a similarity type \( S = \langle I, J, K, \rho, \mu \rangle \) as before.

Let \( a, b \in K_0 \) be as in 2.4, and assume \( A \subseteq B \).

Then, according to 2.4 with \( \pi = \text{id}_A \) we have that \( A \subseteq B \) iff

3.1 (1) for each \( i \in I \) and all \( a_1, \ldots, a_m \in A \) (where \( n = \rho(i) \))

\[ a_1, \ldots, a_m \in R_i \iff \langle a_1, \ldots, a_m \rangle \in S_i. \]

i.e. \( R_i = A^n \cap S_i \) \((*)\)

(2) for each \( j \in J \) and all \( a_1, \ldots, a_m \in A \) (where \( m = \mu(j) \))

\[ f_j(a_1, \ldots, a_m) = g_j(a_1, \ldots, a_m). \]

i.e. \( f_j \) is the restriction of \( g_j \) to \( A^n : f_j = g_j \mid A^n \) \((**)*\)

(\( \text{In particular, if } a_1, \ldots, a_m \in A, \text{ then } g_j(a_1, \ldots, a_m) \in A. \))

(3) for each \( k \in K, \) \( d_k = \pi_k \) \((**)(*) \) (\( \text{In particular, } \)

\( d_k \in A.) \)

3.2 Now suppose that \( B \in K_0 \) (as above) and that \( A \) is a subset of \( B \). We ask the question:

what conditions on \( A \) ensure that \( A \) is the domain of a substructure of \( B \)? By the comments above we must have that whenever \( a_1, \ldots, a_m \in A \) then

\[ g_j(a_1, \ldots, a_m) \in A \] (for each \( j \in J \), where \( m = \mu(j) \)) \( - \) i.e. \( A \)

is closed under \( g_j \) \( - \) and for each \( k \in K, d_k \in A. \)

3.3 Theorem

With the notation above, \( A \) is the domain of a substructure of \( B \) if and only if \( A \) is closed under each \( g_j \) (for \( j \in J \)) and contains each \( d_k \) (for \( k \in K \).
Proof: The "only if" is proved in 3.2. For the "if" direction, suppose that A satisfies the stated condition. Then we simply define $R_i$ by (x) (for each $i \in I$), $f_j$ by (x) (for each $j \in J$) and $g_k$ by (x) (for each $k \in K$), thus defining the substructure $\langle A, \{R_i : i \in I\}, \{f_j : j \in J\}, \{g_k : k \in K\} \rangle$ of $\mathcal{L}$.

3.4 Remark
(a) It follows from 3.3 that if $\sigma$ is a purely relational type, i.e. $J = K = \emptyset$, then every subset of the domain of a $\sigma$-structure is the domain of a substructure of that structure.
(b) It of course follows from 3.1 (1), (2) and (3) that a subset of the domain of a $\sigma$-structure can be the domain of at most one substructure.

3.5 Theorem
Let $\mathcal{L} \in K_o$ (in above notation) and let $S$ be any non-empty subset of $B$. Then there exists a smallest substructure $C_\sigma \subseteq \mathcal{L}$ with $S \subseteq \text{dom}(C_\sigma)$. (This means that if $C_\sigma' \subseteq \mathcal{L}$ and $S \subseteq \text{dom}(C_\sigma')$, then $C_\sigma \subseteq C_\sigma'$.) Further, $C_\sigma$ is unique with this property.

Proof:
Let $S_0 := S \cup \{d_k : k \in K\}$ and, for each $n \geq 0$, let $S_{n+1} := S_n \cup \{g_j(a_1, \ldots, a_{\kappa(j)}) : j \in J, a_1, \ldots, a_{\kappa(j)} \in S_n\}$.

Let $S_\omega := \bigcup_{n=0}^\infty S_n$.

Clearly, $d_k \in S_\omega$ for each $k \in K$.
Also, $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n \subseteq S_{n+1} \subseteq \cdots \subseteq S_\omega$. (†).
Further, if \( j \in J \) then \( S_w \) is closed under \( g_j \). For if \( w = \mu(j) \) and \( a_1, \ldots, a_m \in S_w \), then by (\( \dagger \)), there exists \( n \) such that \( a_1, \ldots, a_m \in S_n \). So \( g_j(a_1, \ldots, a_m) \in S_{n+1} \) (by definition of \( S_{n+1} \)). Since \( S_{n+1} \subseteq S_w \) (by (\( \dagger \))) we have \( g_j(a_1, \ldots, a_m) \in S_w \), as required.

Thus, by 3.3, \( S_w \) is the domain of a substructure, \( \mathcal{C} \) say, of \( \mathcal{B} \). Since \( S \subseteq S_0 \subseteq S_w \) we have \( S \subseteq \text{dom}(\mathcal{C}) \).

Now suppose that \( \mathcal{C}' \) is any substructure of \( \mathcal{B} \) with \( S \subseteq \text{dom}(\mathcal{C}') \). By 3.1(3) we also have \( a_k \in \text{dom}(\mathcal{C}') \) (for each \( k \in K \)), so \( S_0 \subseteq \text{dom}(\mathcal{C}') \). We now prove by induction that for all \( n \), \( S_n \subseteq \text{dom}(\mathcal{C}') \).

So suppose (for the inductive step) that \( S_n \subseteq \text{dom}(\mathcal{C}') \). Since \( \text{dom}(\mathcal{C}') \) is closed under \( g_j \) (for each \( j \in J \)) by 3.1(2), it follows that \( g_j(a_1, \ldots, a_m) \in \text{dom}(\mathcal{C}') \) for each \( a_1, \ldots, a_m \in S_n \) (and each \( j \in J \)). Thus \( S_{n+1} \subseteq \text{dom}(\mathcal{C}') \) as required.

Thus \( \text{dom}(\mathcal{C}) = S_w = \bigcup_{n=0}^{\infty} S_n \subseteq \text{dom}(\mathcal{C}') \). The required result now follows from 3.6 below.

\[ \square \]

3.6 Exercise
Let \( \mathcal{C}, \mathcal{C}', \mathcal{B} \in \mathcal{K} \). Suppose that \( \mathcal{C} \subseteq \mathcal{B} \), \( \mathcal{C}' \subseteq \mathcal{B} \) and \( \text{dom}(\mathcal{C}) \subseteq \text{dom}(\mathcal{C}') \). Then \( \mathcal{C} \subseteq \mathcal{C}' \).

3.7 Definition
A similarity type \( \sigma = \langle I, J, K, \prec, \mu, \nu \rangle \) is called countable if \( I, J, K \) are each countable sets.

3.9 Exercise (Downward Löwenheim-Skolem Theorem: weak form.)
By inspecting the proof of 3.5, prove that if
\( \sigma \) is countable and \( S \) is countable, then \( \text{dom}(\alpha) \) is countable.

3.9 Exercise

Let \( \sigma, \tau, \xi \in K_\sigma \) and suppose that \( \pi : \sigma \rightarrow \tau \) and \( \gamma : \xi \rightarrow \xi \). Prove that \( \gamma \circ \pi : \sigma \rightarrow \xi \).

3.10 Exercise

Prove that any embedding can be decomposed into an isomorphism and identity embedding. I.e. show that if \( \sigma, \tau \in K_\sigma \) and \( \pi : \sigma \rightarrow \tau \), then there exists \( \sigma^* \in K_\sigma \) such that \( \sigma^* \leq \tau \) and \( \pi : \sigma \cong \sigma^* \).