15. A preservation theorem.

Let $L$ be any language. As promised earlier we are now going to prove a converse of 6.4.

15.1 Theorem

Let $\Theta$ be an $L$-sentence. Then the following are equivalent.

1. For all $L$-structures $A$, $B$, if $A \models \Theta$ and $A \subseteq B$, then $B \models \Theta$.
2. $\Theta$ is logically equivalent to an existential $L$-sentence.

Proof.

(2) $\Rightarrow$ (1): Suppose $\models (\Theta \iff E)$ where $E$ is an existential $L$-sentence. To prove (1), let $A$, $B$ be $L$-structures with $A \subseteq B$ and suppose $A \models \Theta$. Then $A \models E$, hence $B \models E$ by 6.4. So $B \models \Theta$.

(1) $\Rightarrow$ (2): Assume $\Theta$ has property (1). Define $\Sigma := \{ \phi : \phi$ is logically equivalent to an existential sentence and $A \models \phi \}$. We aim to show that $\Sigma \models \Theta$. For then, by 10.6, there exists $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \Theta$.

15.1.1 Exercise

Prove that if $\phi_1, \phi_2 \in \Sigma$, then $(\phi_1 \land \phi_2) \in \Sigma$.

Thus, if $\Sigma_0 = \{ \phi_1, \ldots, \phi_n \}$ then $(\bigwedge_{i=1}^n \phi_i) \in \Sigma$, and (since $\Sigma_0 \models \Theta$) we have $(\bigwedge_{i=1}^n \phi_i) \models \Theta$, and so
\( \vdash (\bigwedge_{i=1}^n \phi_i) \to \theta. \)

However, by definition of \( \Sigma \), \( \Sigma = (\bigwedge_{i=1}^n \phi_i) \), i.e.
\( \vdash \theta \to (\bigwedge_{i=1}^n \phi_i) \). Hence \( \theta \) is logically equivalent to \( (\bigwedge_{i=1}^n \phi_i) \), which (since it is in \( \Sigma \)) is logically equivalent to an existential sentence of \( L \). So \( \theta \) is too.

To show that \( \Sigma \models \theta \), let \( \mathcal{L} \models \Sigma \). We must show that \( \mathcal{L} \models \theta \).

To this end we let \( T := \{ \neg \phi : \phi \) an existential sentence of \( L \) such that \( \mathcal{L} \models \neg \phi \} \).

Claim: \( T \cup \{ \theta \} \) is satisfiable.

Proof of claim: If false, \( T \models \neg \theta \) so by 10.6,
\( \{ \neg \phi_1, \ldots, \neg \phi_m \} \models \neg \theta \) for some \( \neg \phi_1, \ldots, \neg \phi_m \in T \).

Thus \( (\bigwedge_{i=1}^n \neg \phi_i) \models \neg \theta \), so by taking the contrapositive we have \( \theta \models (\bigwedge_{i=1}^n \neg \phi_i) \) so
\( \theta \models (\bigwedge_{i=1}^n \phi_i) \). Now (examine) a disjunction of existential sentences of \( L \) is logically equivalent to an existential sentence of \( L \). Hence \( (\bigwedge_{i=1}^n \phi_i) \in \Sigma \).

But \( \mathcal{L} \models \Sigma \), so \( \mathcal{L} \models (\bigwedge_{i=1}^n \phi_i) \). Therefore there is some \( i_0 \in \{1, \ldots, m \} \) such that \( \mathcal{L} \models \phi_{i_0} \). But \( \neg \phi_{i_0} \in T \) and no \( \mathcal{L} \models \neg \phi_{i_0} \) - contradiction. \( \square \) claim.
Thus there exists a model $G \models T \cup \{\Theta\}$.

Now if $\phi$ is any existential sentence of $L$ such that $G \models \phi$, we must have $L \models \phi$, because if $L \models \neg \phi$, then $\neg \phi \in T$ (by definition of $T$) and hence $G \models \neg \phi$ — contradiction. So we may invoke 14.6: there exists an $L$-structure $L'$ with $G \subseteq L'$ and an elementary embedding $\pi : L \to L'$.

We now use our hypothesis (1) to conclude that $L' \models \Theta$ (since $G \models \Theta$ and $G \subseteq L'$). But $L \equiv L'$, so $L \models \Theta$ was required.

15.2 Another application of the method of 14.6

Suppose that $G$, $L$ are $L$-structures with $G \subseteq L$. Consider the set $\Sigma := C(\text{Diag}(G)) \cup \text{Diag}(L)$ of sentences of $L(L)$, where we use the same constant symbols $C_a$ for $a \in A$ that are used in $L(L)$ (via the inclusion $A \subseteq B$), when formulating $C(\text{Diag}(G))$. Suppose that $L \models \Sigma$, and let $L := L^{' \uparrow L}$.

Then, after renaming elements of $C := \text{dom}(L)$ we may suppose that $G \subseteq L \subseteq L$ and that $G \subseteq L$ (by 14.3 and 14.4). But now consider an existential formula $\phi \in F_m(L)$. Suppose $a_1, \ldots, a_n \in A$ are such that $L \models \phi[a_1, \ldots, a_n]$. Then $L \models \phi[a_1, \ldots, a_n]$ (by 6.4). But since $G \subseteq L$, we have $G \models \phi[a_1, \ldots, a_n]$. So we have shown that existential formulas are preserved from $L$ to $G$ — which is certainly not the case in general!! (E.g. consider the additive group of even integers as a substructure of the additive group of integers — see 6.5.)

15.2.1 Exercise*

Find out what has gone wrong here, and formulate and prove the right theorem for the existence of $L$ as above.