15. A preservation theorem.

Let $\mathcal{L}$ be any language. As promised earlier we are now going to prove a converse of 6.4

15.1 Theorem

Let θ be an $\mathcal{L}$-sentence. Then the following are equivalent.

(1) for all $\mathcal{L}$-structures $\mathfrak{A}, \mathfrak{B}$, if $\mathfrak{A} \models \theta$ and $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{B} \models \theta$.

(2) θ is logically equivalent to an existential $\mathcal{L}$-sentence.

Proof.

(2) $\Rightarrow$ (1): Suppose $\models (\theta \Leftrightarrow E)$ where E is an existential $\mathcal{L}$-sentence. To prove (1), let $\mathfrak{A}, \mathfrak{B}$ be $\mathcal{L}$-structures with $\mathfrak{A} \subseteq \mathfrak{B}$ and suppose $\mathfrak{A} \models \theta$. Then $\mathfrak{A} \models E$, hence $\mathfrak{B} \models E$ by 6.4. So $\mathfrak{B} \models \theta$.

(1) $\Rightarrow$ (2): Assume θ has property (1). Define

$\Sigma := \{\phi : \phi$ is logically equivalent to an existential sentence and $\theta \models \phi\}$.

We aim to show that $\Sigma \models \theta$. For then, by 10.6, there exists $\Sigma_0 \subseteq_{fin} \Sigma$ such that $\Sigma_0 \models \theta$.

15.1.1 Exercise

Prove that if $\phi_1, \phi_2 \in \Sigma$, then $(\phi_1 \wedge \phi_2) \in \Sigma$.

Thus, if $\Sigma_0 = \{\phi_1, \dots, \phi_\ell\}$ then $(\bigwedge_{i=1}^{\ell} \phi_i) \in \Sigma$

and (since $\Sigma_0 \models \theta$) we have $(\bigwedge_{i=1}^{\ell} \phi_i) \models \theta$, and so

$\models (\bigwedge_{i=1}^{\ell} \phi_i) \to \theta.$

However, by definition of Σ, $\theta \models (\bigwedge_{i=1}^{\ell} \phi_i)$, i.e. $\models \theta \to (\bigwedge_{i=1}^{\ell} \phi_i)$. Hence θ is logically equivalent to $(\bigwedge_{i=1}^{\ell} \phi_i)$ which (since it is in Σ) is logically equivalent to an existential sentence of $\mathcal{L}$. So θ is too.

To show that $\Sigma \models \theta$, let $\mathcal{B} \models \Sigma$. We must show that $\mathcal{B} \models \theta$.
To this end we let $T := \{\neg\phi : \phi$ an existential sentence of $\mathcal{L}$ such that $\mathcal{B} \models \neg\phi\}$.

<u>Claim</u>: $T \cup \{\theta\}$ is satisfiable

<u>Proof of claim</u>: If false, $T \models \neg\theta$ so by 10.6,

$\{\neg\phi_1, \ldots, \neg\phi_m\} \models \neg\theta$ for some $\neg\phi_1, \ldots, \neg\phi_m \in T$.

Thus $\left(\bigwedge_{i=1}^{m} \neg\phi_i\right) \models \neg\theta$, so by taking the contrapositive we have $\theta \models \neg(\bigwedge_{i=1}^{m} \neg\phi_i)$ so $\theta \models (\bigvee_{i=1}^{m} \phi_i)$. Now (exercise) a disjunction of existential sentences of $\mathcal{L}$ is logically equivalent to an existential sentence of $\mathcal{L}$. Hence $(\bigvee_{i=1}^{m} \phi_i) \in \Sigma$. But $\mathcal{B} \models \Sigma$, so $\mathcal{B} \models (\bigvee_{i=1}^{m} \phi_i)$. Therefore there is some $i_0 \in \{1, \ldots, m\}$ such that $\mathcal{B} \models \phi_{i_0}$. But $\neg\phi_{i_0} \in T$ and so $\mathcal{B} \models \neg\phi_{i_0}$ – contradiction.

$\square$ claim.

Thus there exists a model $\mathfrak{C} \models T \cup \{\theta\}$. Now if ϕ is any existential sentence of $\mathcal{L}$ such that $\mathfrak{C} \models \phi$, we must have $\mathfrak{B} \models \phi$, because if $\mathfrak{B} \models \neg\phi$ then $\neg\phi \in T$ (by definition of T) and hence $\mathfrak{C} \models \neg\phi$ – contradiction. So we may invoke 14.6: there exists an $\mathcal{L}$-structure $\mathfrak{B}'$ with $\mathfrak{C} \subseteq \mathfrak{B}'$ and an elementary embedding $\pi : \mathfrak{B} \to \mathfrak{B}'$. We now use our hypothesis (1) to conclude that $\mathfrak{B}' \models \theta$ (since $\mathfrak{C} \models \theta$ and $\mathfrak{C} \subseteq \mathfrak{B}'$). But $\mathfrak{B} \equiv \mathfrak{B}'$, so $\mathfrak{B} \models \theta$ as required.

$\square$

15.2 <u>Another application of the method of 14.6</u>

Suppose that $\mathfrak{A}, \mathfrak{B}$ are $\mathcal{L}$-structures with $\mathfrak{A} \subseteq \mathfrak{B}$. Consider the set $\Sigma := \mathrm{CDiag}(\mathfrak{A}) \cup \mathcal{D}iag(\mathfrak{B})$ of sentences of $\mathcal{L}(\mathfrak{B})$, where we use the <u>same</u> constant symbols c_a for $a \in A$ that are used in $\mathcal{L}(\mathfrak{B})$ (via the inclusion $A \subseteq B$), when formulating $\mathrm{CDiag}(\mathfrak{A})$. Suppose that $\mathbb{L}' \models \Sigma$, and let $\mathbb{L} := \mathbb{L}' \restriction \mathcal{L}$. Then, after renaming elements of C ($:= \mathrm{dom}(\mathbb{L})$) we may suppose that $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathbb{L}$ and that $\mathfrak{A} \preccurlyeq \mathbb{L}$ (by 14.3 and 14.4). But now consider an existential formula $\phi \in F_n(\mathcal{L})$. Suppose $a_1, \ldots, a_n \in A$ are such that $\mathfrak{B} \models \phi[a_1, \ldots, a_n]$. Then $\mathbb{L} \models \phi[a_1, \ldots, a_n]$ (by 6.4). But since $\mathfrak{A} \preccurlyeq \mathbb{L}$, we have $\mathfrak{A} \models \phi[a_1, \ldots, a_n]$. So we have shown that existential formulas are preserved from $\mathfrak{B}$ to $\mathfrak{A}$ – which is certainly not the case in general!! (E.g. consider the additive group of even integers as a substructure of the additive group of integers – see 6.5.)

15.2.1 <u>Exercise*</u>

Find out what has gone wrong here, and formulate and prove the right theorem for the existence of $\mathbb{L}$ as above.