2. Structures - the precise definition.

2.1 Definition

(a) A *similarity type* consists of three sets I, J, K (some, or all, of which may be empty) and two functions $\rho : I \to \mathbb{N}_{>0}$, $\mu : J \to \mathbb{N}_{>0}$.

(b) Let $\sigma = \langle I, J, K, \rho, \mu \rangle$ be a similarity type. Then a *structure of similarity type* σ, or just *σ-structure* for short, is an object of the form

$$\mathfrak{A} = \langle A ; \{R_i\}_{i \in I} ; \{f_j\}_{j \in J} ; \{e_k\}_{k \in K} \rangle$$

where:

(i) A is a non-empty set, called the *domain* of $\mathfrak{A}$ ($A = \mathrm{dom}(\mathfrak{A})$);

(ii) for each $i \in I$, $R_i \subseteq A^n$ $(= \underbrace{A \times A \times \cdots \times A}_{n})$ where $n = \rho(i)$. We say that R_i is an *n-ary relation* on A;

(iii) for each $j \in J$, $f_j : A^m \to A$ where $m = \mu(j)$. We say that f_j is an *m-ary function* on A;

(iv) for each $k \in K$, $e_k \in A$. We say that e_k is a *distinguished element* of A.

2.2 Examples

Let us look at 1.1 (a) and (b): $\langle \mathbb{R} ; + ; 0 \rangle$, $\langle \mathbb{R} ; \cdot ; 1 \rangle$.
These are structures with $I = \emptyset$, $J = \{1\}$, $K = \{1\}$ and $\mu(1) = 2$.
In 1.1 (a) we have f_1 is $+$, e_1 is 0 and $A = \mathbb{R}$.
In 1.1 (b) we have f_1 is $\bullet$, e_1 is 1 and $A = \mathbb{R}$.

Now consider 1.2: $\langle \mathbb{R}; <; +; 0 \rangle$.
Here we have $I = \{1\}$, $J = \{1\}$, $K = \{1\}$, $\rho(1) = 2$, $\mu(1) = 2$.
Then $A = \mathbb{R}$, R_1 is $<$ (considered as the subset $\{\langle a, b\rangle : a \in \mathbb{R}, b \in \mathbb{R}, a < b\}$ of $\mathbb{R}^2$), f_1 is $+$, and e_1 is 0.

2.3 <u>Definition</u>
Let σ be a similarity type. Then K_σ denotes the class of all structures of similarity type σ.

2.4 <u>Definition</u> (<u>of isomorphism</u>).
Let σ be a similarity type and suppose that $\mathfrak{A}, \mathfrak{B} \in K_\sigma$.
Say $\mathfrak{A} = \langle A; \{R_i\}_{i\in I}; \{f_j\}_{j\in J}; \{e_k\}_{k\in K}\rangle$
and
$$\mathfrak{B} = \langle B; \{S_i\}_{i\in I}; \{g_j\}_{j\in J}; \{d_k\}_{k\in K}\rangle.$$

Let $\pi : A \to B$ be a function. Then we say that π is an <u>isomorphism</u> from $\mathfrak{A}$ to $\mathfrak{B}$ if

(a) π is one-one and onto (i.e. a bijection from A to B);

(b) for each $i \in I$, if $\rho(i) = n$, then for all $a_1, \dots, a_n \in A$,
$$\langle a_1, \dots, a_n\rangle \in R_i \text{ iff } \langle \pi(a_1), \dots, \pi(a_n)\rangle \in S_i \quad ;$$

(c) for each $j \in J$, if $\mu(j) = m$, then for all $a_1, \dots, a_m \in A$,
$$\pi(f_j(a_1, \dots, a_m)) = g_j(\pi(a_1), \dots, \pi(a_m)) \ ;$$

(d) for each $k \in K$, $\pi(e_k) = d_k$.

We write $\pi : \mathfrak{A} \cong \mathfrak{B}$ to mean π is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. We also write just $\mathfrak{A} \cong \mathfrak{B}$ to mean that there exists some π s.th. $\pi : \mathfrak{A} \cong \mathfrak{B}$, and then we say that <u>$\mathfrak{A}$ is isomorphic to $\mathfrak{B}$</u>.

2.5 <u>Example</u>
Let $\mathfrak{A} = \langle \mathbb{R}; <; +; 0\rangle$, $\mathfrak{B} = \langle \mathbb{R}_{>0}; <; \cdot; 1\rangle$.

Then $\mathfrak{A} \cong \mathfrak{B}$. For consider the function $\pi : \mathbb{R} \to \mathbb{R}_{>0}$ given by $\pi(x) = e^x$ (for $x \in \mathbb{R}$). Then 2.4(a)–(d) are easy to check. E.g. for (c) we have:- for all $a_1, a_2 \in \mathbb{R}$ $\quad \pi(a_1 + a_2) = e^{(a_1+a_2)} = e^{a_1} \cdot e^{a_2} = \pi(a_1) \cdot \pi(a_2)$.

2.6 A non-example is given by $\mathfrak{A} = \langle \mathbb{Q}; < \rangle$, $\mathfrak{B} = \langle \mathbb{R}; < \rangle$. $\mathfrak{A}$ cannot be isomorphic to $\mathfrak{B}$ since there is not even a bijection from $\mathbb{Q}$ (which is countable) to $\mathbb{R}$ (which is not). However, the identity function on $\mathbb{Q}$ (considered as a function from $\mathbb{Q}$ to $\mathbb{R}$) does satisfy all the conditions for being an isomorphism except that it is not onto $\mathbb{R}$. This gives rise to another important notion. (<u>An aside</u>: We shall show later that $Th(\langle \mathbb{Q}; < \rangle) = Th(\langle \mathbb{R}; < \rangle)$, so it is certainly possible for two structures to have the same theory, but not be isomorphic – see 1.11 (2)(c))

2.7 <u>Definition</u>

(a) In the notation of 2.4, if π satisfies (b), (c) and (d) and is one-one (but not necessarily onto), then π is called an <u>embedding</u>. We write $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$ for this.

(b) If $A \subseteq B$ and π is the identity function on A (and also satisfies 2.4 (b), (c) and (d)) then we say that $\mathfrak{A}$ is a <u>substructure</u> of $\mathfrak{B}$, written $\mathfrak{A} \subseteq \mathfrak{B}$.

2.8 <u>Examples</u>

- $\langle \mathbb{R}; +, \cdot; 0 \rangle \subseteq \langle \mathbb{C}; +, \cdot; 0 \rangle$; $\langle \mathbb{Q}; <; +; 0 \rangle \subseteq \langle \mathbb{R}; <; +; 0 \rangle$;

but $\langle \mathbb{R}; +, \cdot; 0 \rangle \not\subseteq \langle \mathbb{C}; +, \cdot; 1 \rangle$ because 2.4(d) fails.

"is not a substructure of"