14.6 Lemma

Let $\mathfrak{A}, \mathfrak{B}$ be any $\mathcal{L}$-structures. Then the following are equivalent.

(1) for every existential sentence ϕ of $\mathcal{L}$, if $\mathfrak{A} \models \phi$ then $\mathfrak{B} \models \phi$;

(2) there exists an $\mathcal{L}$-structure $\mathfrak{B}'$ such that $\mathfrak{A} \subseteq \mathfrak{B}'$ and an elementary embedding $\pi : \mathfrak{B} \to \mathfrak{B}'$ (so, in particular, $\mathfrak{B} \equiv \mathfrak{B}'$).

Proof.

(2) $\Rightarrow$ (1): Suppose ϕ is existential and $\mathfrak{A} \models \phi$. By 6.4 we have $\mathfrak{B}' \models \phi$. So $\mathfrak{B} \models \phi$ since $\mathfrak{B} \equiv \mathfrak{B}'$.

(1) $\Rightarrow$ (2): We use the method of diagrams. We choose the new constant symbols $\{c_a : a \in A\}$ and $\{c_b : b \in B\}$ to be disjoint (as well as different from the constant symbols in $\mathcal{L}$).

Let Σ be the following set of sentences (in the language containing all the constant symbols):-

$$\mathrm{Diag}(\mathfrak{A}) \cup \mathrm{C}\,\mathrm{Diag}(\mathfrak{B}).$$

It is sufficient to show that Σ has a model, $\mathfrak{B}^*$ say. For by 14.4, the function $\pi : B \to B^* : b \mapsto c_b^{\mathfrak{B}^*}$ is an elementary embedding from $\mathfrak{B}$ to $\mathfrak{B}^* \restriction \mathcal{L}$, and the function $\pi' : A \to B^* ; c_a \mapsto c_a^{\mathfrak{B}^*}$ is an embedding from $\mathfrak{A}$ to $\mathfrak{B}^* \restriction \mathcal{L}$. We may assume that the latter embedding is the identity (so that $\mathfrak{A} \subseteq \mathfrak{B}'$) by simply renaming each element $\pi'(a)$ of B^* as a. Thus $\pi : \mathfrak{B} \preccurlyeq \mathfrak{B}'$ and $\mathfrak{A} \subseteq \mathfrak{B}'$, where $\mathfrak{B}' := \mathfrak{B}^* \restriction \mathcal{L}$.

Thus, by the Compactness Theorem, it is sufficient to show that Σ is finitely satisfiable.

So let $\Sigma_0 \underset{fin}{\subseteq} \Sigma$.

Then there are $\phi_1, \ldots, \phi_\ell \in \mathrm{Diag}(\mathfrak{A})$ such that $\Sigma_0 \subseteq \{\phi_1, \ldots, \phi_\ell\} \cup C\mathcal{D}iag(\mathfrak{B})$.

Let $\phi = \left(\bigwedge_{i=1}^{\ell} \phi_i\right)$. Then $\phi \in \mathrm{Diag}(\mathfrak{A})$ and we must show that $C\mathcal{D}iag(\mathfrak{B}) \cup \{\phi\}$ has a model. So suppose that it does not. Then $C\mathcal{D}iag(\mathfrak{B}) \models \neg\phi$.
Now we may write ϕ in the form $\psi(c_{a_1}, \ldots, c_{a_n})$ where $\psi \in F_n(\mathcal{L})$ (for some $n \geq 0$: see 4.2), and ψ is quantifier-free.
Thus $\quad C\mathcal{D}iag(\mathfrak{B}) \models \neg\psi(c_{a_1}, \ldots, c_{a_n})$.

But we chose our new constant symbols so that $c_{a_1}, \ldots, c_{a_n}$ do not occur in any formula in $C\mathcal{D}iag(\mathfrak{B})$.
Thus by 14.5

$$C\mathcal{D}iag(\mathfrak{B}) \models \forall v_1 \cdots \forall v_n \neg\psi, \quad \text{i.e.}$$
$$C\mathcal{D}iag(\mathfrak{B}) \models \neg\exists v_1 \cdots \exists v_n \psi \text{ ---- } (*).$$

However, since $\phi \in \mathcal{D}iag(\mathfrak{A})$, we have $\mathfrak{A}^+ \models \phi$, i.e. $\mathfrak{A}^+ \models \psi(c_{a_1}, \ldots, c_{a_n})$. So $\mathfrak{A} \models \psi[a_1, \ldots, a_n]$ (see 4.2).
Hence $\mathfrak{A} \models \exists v_1 \cdots \exists v_n \psi$.
But $\exists v_1 \cdots \exists v_n \psi$ is an existential sentence of $\mathcal{L}$, so by our hypothesis (1), $\mathfrak{B} \models \exists v_1 \cdots \exists v_n \psi$. So $\exists v_1 \cdots \exists v_n \psi \in C\mathcal{D}iag(\mathfrak{B})$, which contradicts $(*)$ (since $C\mathcal{D}iag(\mathfrak{B})$ is certainly satisfiable).
Thus ϕ, and hence Σ_0, does have a model, and we are done.

$\square$