14. The method of diagrams.

Let $\mathcal{L}$ be an arbitrary language and $\mathfrak{A}$ an arbitrary $\mathcal{L}$-structure. We form a new language, denoted $\mathcal{L}(\mathfrak{A})$, by adding a new constant symbol c_a for each element $a \in A$. We may consider $\mathfrak{A}$ as an $\mathcal{L}(\mathfrak{A})$-structure by interpreting c_a as a. We denote this $\mathcal{L}(\mathfrak{A})$-structure by $\mathfrak{A}^+$.

14.1 Definitions

(1) By the diagram of $\mathfrak{A}$ we mean the set of all quantifier-free $\mathcal{L}(\mathfrak{A})$-sentences that are true in $\mathfrak{A}^+$:

$$\mathcal{D}iag(\mathfrak{A}) := \{\phi : \phi \text{ a QF sentence of } \mathcal{L}(\mathfrak{A}) \text{ s.th. } \mathfrak{A}^+ \models \phi\}.$$

(2) By the complete diagram of $\mathfrak{A}$ we mean the set of all $\mathcal{L}(\mathfrak{A})$-sentences that are true in $\mathfrak{A}^+$:

$$C\mathcal{D}iag(\mathfrak{A}) := \{\phi : \phi \text{ an } \mathcal{L}(\mathfrak{A})\text{-sentence and } \mathfrak{A}^+ \models \phi\}.$$

14.2 Remark

An $\mathcal{L}(\mathfrak{A})$-sentence, ϕ say, has the form $\psi(c_{a_1}, \dots, c_{a_n})$ for some $n \geq 0$, some $\psi \in F_n(\mathcal{L})$ and some $a_1, \dots, a_n \in A$. (Proof: an easy induction on ϕ.). Then $\phi \in C\mathcal{D}iag(\mathfrak{A})$ (resp: $\mathcal{D}iag(\mathfrak{A})$) $\Leftrightarrow$ $\mathfrak{A} \models \psi[a_1, \dots, a_n]$ (resp: and ψ is QF).

Similarly, if $\mathfrak{B}$ is any $\mathcal{L}(\mathfrak{A})$-structure, (with ϕ, ψ as above),

$$\mathfrak{B} \models \phi \Leftrightarrow \mathfrak{B} \restriction \mathcal{L} \models \psi[c_{a_1}^{\mathfrak{B}}, \dots, c_{a_n}^{\mathfrak{B}}],$$ where $\mathfrak{B} \restriction \mathcal{L}$

denotes the $\mathcal{L}$-structure obtained from $\mathfrak{B}$ by forgetting the interpretation of the new constants. (Thus, if $\mathfrak{B}$ is $\langle B; \{\dots\}_{i\in I}; \{\dots\}_{j\in J}; \{c_k\}_{k\in K\cup A}\rangle$, then $\mathfrak{B}\restriction\mathcal{L}$ is $\langle B; \{\dots\}_{i\in I}; \{\dots\}_{j\in J}; \{c_k\}_{k\in K}\rangle$. Then $\mathfrak{B}\restriction\mathcal{L}$ is an

$\mathcal{L}$-structure called the <u>reduct of $\mathfrak{B}$ to $\mathcal{L}$</u>. (We also call $\mathfrak{B}$ an <u>expansion of $\mathfrak{B}\restriction\mathcal{L}$ to $\mathcal{L}(\mathfrak{A})$</u>.)

14.3 <u>Theorem</u> (The method of diagrams (1))

Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and $\mathfrak{B}$ be an $\mathcal{L}(\mathfrak{A})$-structure such that $\mathfrak{B} \models \mathrm{Diag}(\mathfrak{A})$. Then the function $\pi : A \to B$ defined by $\pi(a) := c_a^{\mathfrak{B}}$ is an embedding from $\mathfrak{A}$ to $\mathfrak{B}\restriction\mathcal{L}$.

<u>Proof.</u>

Suppose P is an n-ary relation symbol of $\mathcal{L}$, $a_1, \dots, a_n \in A$, and $\mathfrak{A} \models P[a_1, \dots, a_n]$. Then by 14.2, $P(c_{a_1}, \dots, c_{a_n}) \in \mathrm{Diag}(\mathfrak{A})$ and hence $\mathfrak{B} \models P(c_{a_1}, \dots, c_{a_n})$. Then $\mathfrak{B} \models P[c_{a_1}^{\mathfrak{B}}, \dots, c_{a_n}^{\mathfrak{B}}]$, i.e. $\mathfrak{B} \models P[\pi(a_1), \dots, \pi(a_n)]$. So $\mathfrak{B}\restriction\mathcal{L} \models P[\pi(a_1), \dots, \pi(a_n)]$.

Similarly, if $\mathfrak{A} \models \neg P[a_1, \dots, a_n]$, then $\neg P(c_{a_1}, \dots, c_{a_n}) \in \mathrm{Diag}(\mathfrak{A})$ and proceed as above to get $\mathfrak{B}\restriction\mathcal{L} \models \neg P[\pi(a_1), \dots, \pi(a_n)]$.

Thus, if $S \subseteq B^n$ is the relation interpreting P in $\mathfrak{B}$, we have shown that for all $a_1, \dots, a_n \in A$, $\langle a_1, \dots, a_n \rangle \in R \iff \langle \pi(a_1), \dots, \pi(a_n) \rangle \in S$, where R interprets P in $\mathfrak{A}$.

Now suppose that F is an n-ary function symbol of $\mathcal{L}$, interpreted by f in $\mathfrak{A}$ and g in $\mathfrak{B}$.

Let $a_1, \dots, a_n \in A$. Say $f(a_1, \dots, a_n) = a$ $(\in A)$. Then the $\mathcal{L}(\mathfrak{A})$ sentence $F(c_{a_1}, \dots, c_{a_n}) \doteq c_a$ is in $\mathrm{Diag}(\mathfrak{A})$ and so is true in $\mathfrak{B}$. It follows that $g(c_{a_1}^{\mathfrak{B}}, \dots, c_{a_n}^{\mathfrak{B}}) = c_a^{\mathfrak{B}}$, i.e. $g(\pi(a_1), \dots, \pi(a_n)) = \pi(a) = \pi(f(a_1, \dots, a_n))$.

Finally, if c_k is a constant symbol of $\mathcal{L}$, then $c_k^{\mathfrak{A}} = a$ for some $a \in A$, so the $\mathcal{L}(\mathfrak{A})$ sentence $c_k \doteq c_a$ is in $\mathrm{Diag}(\mathfrak{A})$. So $\mathfrak{B} \models c_k \doteq c_a$, i.e. $c_k^{\mathfrak{B}} = c_a^{\mathfrak{B}} = \pi(a) = \pi(c_k^{\mathfrak{A}})$.

A similar argument shows that π is one-one. $\square$

14.4 <u>Corollary</u> (The method of diagrams (2).)

With the same notation as in 14.3, if we have $\mathfrak{B} \models CDiag(\mathfrak{A})$ then the function π is an elementary embedding from $\mathfrak{A}$ to $\mathfrak{B} \upharpoonright \mathcal{L}$.

<u>Proof.</u>

Obviously $Diag(\mathfrak{A}) \subseteq CDiag(\mathfrak{A})$ so $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$ by 14.3. Let $\phi \in F_n(\mathcal{L})$ and $a_1, \ldots, a_n \in A$.

Then $\mathfrak{A} \models \phi[a_1, \ldots, a_n] \Rightarrow \mathfrak{A}^+ \models \phi(c_{a_1}, \ldots, c_{a_n})$

$\Rightarrow \phi(c_{a_1}, \ldots, c_{a_n}) \in CDiag(\mathfrak{A}) \Rightarrow \mathfrak{B} \models \phi(c_{a_1}, \ldots, c_{a_n})$

$\Rightarrow \quad \mathfrak{B} \upharpoonright \mathcal{L} \models \phi[c_{a_1}^{\mathfrak{B}}, \ldots, c_{a_n}^{\mathfrak{B}}]$ (by 14.2)

$\Rightarrow \quad \mathfrak{B} \upharpoonright \mathcal{L} \models \phi[\pi(a_1), \ldots, \pi(a_n)]$ (by def. of π).

Thus $\pi : \mathfrak{A} \preccurlyeq \mathfrak{B} \upharpoonright \mathcal{L}$.

$\square$

• Suppose $\mathfrak{A}, \mathfrak{B}$ are $\mathcal{L}$-structures and suppose there exists an embedding from $\mathfrak{A}$ to $\mathfrak{B}$. Then by 6.4 we know that every existential sentence of $\mathcal{L}$ that is true in $\mathfrak{A}$ must be true in $\mathfrak{B}$. The converse is false (e.g. we could have $\mathfrak{A} \equiv \mathfrak{B}$, $\mathfrak{B}$ countable and $\mathfrak{A}$ uncountable). The following lemma (14.6) provides the next best thing. We first require the result contained in the following

14.5 <u>Exercise</u>

Let Σ be an $\mathcal{L}$-theory and let $\phi \in F_n(\mathcal{L})$ (for some $n \geqslant 0$). Let $c_1, \ldots, c_n$ be constant symbols which do not occur in any formula in $\{\phi\} \cup \Sigma$. Assume that $\Sigma \models \phi(c_1, \ldots, c_n)$. Then $\Sigma \models \forall v_1 \cdots \forall v_n \phi$.