13.4 Definition

An n-type \( p \) is called principal if there is some \( \phi \in p \) such that \( \phi \) is principal for \( p \), i.e. for all \( \psi \in p \), \( T \models \forall \eta_1 \ldots \forall \eta_m (\phi \rightarrow \psi) \).

13.5 Exercises

1) There is no clash in terminology with our use of the word 'principal' in section 12. For if \( G \models T \) and \( a_1, \ldots, a_m \in A \) realises the principal n-type \( p \), then \( \phi \) is principal for \( a_1, \ldots, a_m \) (in \( G \)), where \( \phi \) is a principal formula of \( p \).

2) If \( \phi \) and \( \psi \) are both principal for the n-type \( p \), then \( \phi \) and \( \psi \) are \( \text{Em}(T) \)-equivalent.

13.6 Theorem

If \( p \) is a principal n-type (over \( T \)) then \( p \) is realised in every model of \( T \).

Proof.

Let \( G \models T \). Let \( \phi \) be principal for the n-type \( p \). Then since \( \phi \in p \) we have \( T \models \forall \eta_1 \ldots \forall \eta_m \phi \), so for some \( a_1, \ldots, a_m \in A \), \( G \models \phi[a_1, \ldots, a_m] \). Then \( a_1, \ldots, a_m \) realises \( p \) in \( G \) since if \( \psi \in p \), then \( T \models \forall \eta_1 \ldots \forall \eta_m (\phi \rightarrow \psi) \), so in particular \( G \models (\phi \rightarrow \psi)[a_1, \ldots, a_m] \). Since \( G \models \phi[a_1, \ldots, a_m] \) we have \( G \models \psi[a_1, \ldots, a_m] \), as required.

Much deeper is the

13.7 Omitting Types Theorem

Suppose \( \mathcal{L} \) is countable, \( T \) is a complete \( \mathcal{L} \)-theory, \( n \geq 1 \) and the \( p \) is an n-type (over \( T \)) which is not principal. Then there exists a countable model of \( T \).
that omits \( T \).

\textbf{Proof.}

(omitted. But note will be supplied. \( \square \))

We now complete the proof of the Ryll-Nardzewski Theorem (12.2) by showing the following. (See 13.3(4).)

13.8 Theorem

\[ L \] is countable and suppose that \( T \) is a complete \( L \)-theory. Let \( n \geq 1 \) and assume that \( F_n(L)/E_n(T) \) is infinite. Then there exists a non-principal \( n \)-type (over \( T \)).

\textbf{Proof.}

Suppose, for a contradiction, that every \( n \)-type (over \( T \)) is principal.

\textbf{Case 1.} There are only finitely many \( n \)-types (over \( T \)).

Let them be \( \beta_1, \ldots, \beta_n \), and let \( \phi_1, \ldots, \phi_n \) be principal formulas for them (respectively). We shall show that \( |F_n(L)/E_n(T)| \leq 2^n \), contrary to our hypothesis.

To see this, let \( \psi \in F_n(L) \) and suppose \( T \models \exists \psi_1 \ldots \exists \psi_n \psi \). Let \( \varphi \in F_n(L) \) be the \( E_n(T) \)-equivalence class of \( \psi \). Then \( \varphi \) is a partial \( n \)-type (over \( T \)) and hence is contained in some \( n \)-type (over \( T \)) by 13.3(1). Say \( \varphi \in \beta_{i_1} \), where \( 1 \leq j_i \leq N \). Then \( \psi \in \beta_{i_1} \), so \( T \models \exists \psi_1 \ldots \exists \psi_n (\phi_{j_1} \rightarrow \psi) \).

Now consider the formula \( \psi' := (\psi \lor \neg \phi_{j_1}) \).

If \( T \models \exists \psi_1 \ldots \exists \psi_n \psi' \), then by repeating the above argument, there exists \( j_2 \) such that \( T \models \forall \psi_1 \ldots \forall \psi_n (\phi_{j_2} \rightarrow \psi') \).
Clearly \( j_1 \neq j_2 \), for otherwise \( T \models \forall v_1 \ldots \forall v_n \left( \phi_{1_{j_1}} \rightarrow \psi_{1_{j_1}} \right) \) which contradicts the fact that \( T \models \exists v_1 \ldots \exists v_n \phi_{1_{j_1}} \).

Now consider the formula \( \psi_3'' : = \left( \psi_3 \land \exists v_1 \ldots \exists v_n \phi_{1_{j_2}} \right) \).

If \( T \models \exists v_1 \ldots \exists v_n \psi_3'' \), then we repeat the argument above to find \( j_3 \) (with \( 1 \leq j_3 \leq n \)) such that
\[
T \models \forall v_1 \ldots \forall v_n \left( \phi_{1_{j_3}} \rightarrow \psi_3'' \right)
\]
and, once again, we must have \( j_2 \neq j_3 \), and \( j_3 \neq j_2 \). Continuing in this way we must eventually run out of \( j_3 \)'s. Say this happens after the \( k \)th stage. This means that we have found distinct \( j_1 \ldots j_k \) (all lying between \( 1 \) and \( n \)) such that

13.8.1 \( T \models \forall v_1 \ldots \forall v_n \left( \phi_{1_{j_i}} \rightarrow \psi \right) \) for each \( i = 1, \ldots, k \), and

13.8.2 \( T \models \exists v_1 \ldots \exists v_n \left( \psi \land \forall v_1 \ldots \forall v_n \phi_{1_{j_1}} \ldots \phi_{1_{j_k}} \right) \).

Now 13.8.1 is equivalent to
\[
T \models \forall v_1 \ldots \forall v_n \left( \bigvee_{k=1}^{N} \phi_{1_{j_k}} \right) \rightarrow \psi,
\]
and 13.8.2 is equivalent to
\[
T \models \forall v_1 \ldots \forall v_n \left( \psi \rightarrow \left( \bigvee_{k=1}^{N} \phi_{1_{j_k}} \right) \right).
\]

So \( \psi \) is \( E_n(T) \)-equivalent to \( \bigvee_{i \in S} \phi_{1_{j_i}} \) for some non-empty subset \( S \) of \( \{1, \ldots, N\} \).

We assume that \( T \models \exists v_1 \ldots \exists v_n \psi \). If this is not the case, then \( \psi \) is \( E_n(T) \)-equivalent to, say, \( \neg v_1 \equiv v_1 \). So the number of \( E_n(T) \)-equivalence classes is at most the number of subsets of \( \{1, \ldots, N\} \). Thus \( |E_n(T)/E_n(T)| \leq 2^N \) as required.

Case 2. There are infinitely many \( n \)-types (over \( T \)).

Let \( \phi_1, \phi_2, \ldots, \phi_n \) be distinct \( n \)-types (over \( T \))
and (since we are assuming that they are all principal)
let \( \phi_1, \phi_2, \ldots, \phi_n \) be principal formulas for them.
Now if \( i \neq j \) then \( T \models \exists \forall i : \exists \forall n ( \phi_i \land \phi_j ) \) \( \iff (*) \).

For if \( i \neq j \), then \( \phi_i \neq \phi_j \) so there is some formula \( \psi \in \psi_i \) such that \( \psi \notin \psi_j \), so \( \forall \psi \in \psi_j \). But then
\[
T \models \forall \psi_i : \forall \psi_n ( \phi_i \to \psi ) \quad \text{and} \quad T \models \forall \psi_i : \forall \psi_n ( \phi_j \to \neg \psi )
\]
which implies \( T \models \exists \forall i : \exists \forall n ( \phi_i \land \phi_j ) \), as required.

(This argument also shows that there are, indeed, at most countably many \( n \)-types (over \( T \)).)

Now let \( q := \{ ( \bigwedge_{i=0}^{s} \phi_i ) : s \leq 3, \text{finite} \} \).

Clearly \( q \) is closed under conjunction, and
for all \( s \leq 3 \), \( T \models \exists \forall i : \exists \forall n ( \bigwedge_{i=0}^{s} \phi_i ) \) because

otherwise we would have \( T \models \forall \forall i : \forall \forall n ( \bigvee_{i=0}^{s} \phi_i ) \). But

if we choose \( j_0 \in \{1, 2, \ldots, 3\} \setminus d \), then certainly
\[
T \models \exists \forall i : \exists \forall n ( \phi_{j_0} \land ( \bigvee_{i=0}^{s} \phi_i ) ) , \quad \text{so} \quad T \models \exists \forall i : \exists \forall n ( \bigvee_{i=0}^{s} \phi_i )
\]

so \( T \models \bigvee_{i=0}^{s} \exists \forall i : \exists \forall n ( \phi_{j_0} \land \phi_i ) \), which clearly contradicts \( (*) \).

This shows that \( q \) is a partial \( n \)-type (over \( T \)), and hence, by 13.3(i), \( q \models p \) for some \( n \)-type \( p \) (over \( T \)).

But \( p = p_{i_0} \) for some \( i_0 \), so \( \phi_{i_0} \in p_{i_0} \). But

\[
\neg \phi_{i_0} \in q \subseteq p_{i_0} , \quad \text{so} \quad ( \phi_{i_0} \land \neg \phi_{i_0} ) \in p_{i_0} - \text{contradiction}.
\]

So we have shown that there must be at least one non-principal \( n \)-type (over \( T \)).
13.9 Exercise

Let $T = Th(<\mathbb{Z}^*; +; 0, 1>)$. Prove that there are uncountably many $1$-types (over $T$).

13.10 Exercise

Let $T$ be $\omega$-categorical and let $\mathfrak{A} \models T$. For $S \subseteq A$, denote by $\langle S \rangle$ the smallest substructure of $\mathfrak{A}$ containing the set $S$ in its domain (see 3.5).

Prove that if $S$ is finite then so is $\langle S \rangle$. 