13.4 <u>Definition</u>

An n-type p is called <u>principal</u> if there is some $\phi \in p$ such that ϕ is <u>principal for</u> p, i.e. for all $\psi \in p$, $T \models \forall v_1 \ldots \forall v_n (\phi \to \psi)$.

13.5 <u>Exercises</u>

(1) There is no clash in terminology with our use of the word 'principal' in section 12. For if $\mathfrak{A} \models T$ and $a_1, \ldots, a_n \in A$ realises the principal n-type p, then ϕ is principal for $a_1, \ldots, a_n$ (in $\mathfrak{A}$), where ϕ is a principal formula of p.

(2) If ϕ and ψ are both principal for the n-type p, then ϕ and ψ are $E_n(T)$-equivalent.

13.6 <u>Theorem</u>

If p is a principal n-type (over T) then p is realised in every model of T.

<u>Proof.</u>

Let $\mathfrak{A} \models T$. Let ϕ be principal for the n-type p. Then since $\phi \in p$ we have $T \models \exists v_1 \ldots \exists v_n \phi$, so for some $a_1, \ldots, a_n \in A$, $\mathfrak{A} \models \phi[a_1, \ldots, a_n]$. Then $a_1, \ldots, a_n$ realises p in $\mathfrak{A}$ since if $\psi \in p$, then $T \models \forall v_1 \ldots \forall v_n (\phi \to \psi)$, so in particular $\mathfrak{A} \models (\phi \to \psi)[a_1, \ldots, a_n]$. Since $\mathfrak{A} \models \phi[a_1, \ldots, a_n]$ we have $\mathfrak{A} \models \psi[a_1, \ldots, a_n]$, as required.

$\square$

Much deeper is the

13.7 <u>Omitting Types Theorem</u>

Suppose $\mathcal{L}$ is countable, T is a complete $\mathcal{L}$-theory, $n \geq 1$ and the p is an n-type (over T) which is not principal. Then there exists a countable model of T

that omits T.

Proof.

Omitted! But notes will be supplied.

$\square$

We now complete the proof of the Ryll-Nardzewski Theorem (12.2) by showing the following. (See 13.3(4).)

13.8 Theorem

Let $\mathcal{L}$ be countable and suppose the T is a complete $\mathcal{L}$-theory. Let $n \geq 1$ and assume that $F_n(\mathcal{L})/E_n(T)$ is infinite. Then there exists a non-principal n-type (over T).

Proof.

Suppose, for a contradiction, that every n-type (over T) is principal.

Case 1 There are only finitely many n-types (over T).

Let them be $p_1, \ldots, p_N$ and let $\phi_1, \ldots, \phi_N$ be principal formulas for them (respectively). We shall show that $\left| F_n(\mathcal{L})/E_n(T) \right| \leq 2^N$, contrary to our hypothesis.

To see this, let $\psi \in F_n(\mathcal{L})$ and suppose $T \models \exists v_1 \ldots \exists v_n \psi$. Let $q \subseteq F_n(\mathcal{L})$ be the $E_n(T)$-equivalence class of ψ. Then q is a partial n-type (over T) and hence is contained in some n-type (over T) by 13.3(1). Say $q \subseteq p_{j_1}$, where $1 \leq j_1 \leq N$. Then $\psi \in p_{j_1}$, so $T \models \forall v_1 \ldots \forall v_n (\phi_{j_1} \to \psi)$.

Now consider the formula $\psi' := (\psi \wedge \neg \phi_{j_1})$.

If $T \models \exists v_1 \ldots \exists v_n \psi'$, then by repeating the above argument, there exists j_2 such that $T \models \forall v_1 \ldots \forall v_n (\phi_{j_2} \to \psi')$.

Clearly $j_1 \neq j_2$, for otherwise $T \models \forall v_1 \cdots \forall v_n (\phi_{j_1} \to \neg \phi_{j_1})$ which contradicts the fact that $T \models \exists v_1 \cdots \exists v_n \phi_{j_1}$.
Now consider the formula $\psi'' := (\psi \wedge \neg\phi_{j_1} \wedge \neg\phi_{j_2})$.
If $T \models \exists v_1 \cdots \exists v_n \psi''$, then we repeat the argument above to find j_3 (with $1 \leq j_3 \leq N$) such that $T \models \forall v_1 \ldots \forall v_n (\phi_{j_3} \to \psi'')$ and, once again, we must have $j_3 \neq j_1$ and $j_3 \neq j_2$. Continuing in this way we must eventually run out of j_i's. Say this happens after the k'th stage. This means that we have found distinct $j_1, \ldots, j_k$ (all lying between 1 and N) such that

13.8.1 $\quad T \models \forall v_1 \cdots \forall v_n (\phi_{j_i} \to \psi)$ for each $i = 1, .., k$, and

13.8.2 $\quad T \models \neg \exists v_1 \cdots \exists v_n (\psi \wedge \neg\phi_{j_1} \wedge \neg\phi_{j_2} \wedge \cdots \wedge \neg\phi_{j_k})$.

Now 13.8.1 is equivalent to

$$T \models \forall v_1 \cdots \forall v_n \left(\left(\bigvee_{i=1}^{k} \phi_{j_i} \right) \to \psi \right),$$

and 13.8.2 is equivalent to

$$T \models \forall v_1 \cdots \forall v_n \left(\psi \to \left(\bigvee_{i=1}^{k} \phi_{j_i} \right) \right).$$

So ψ is $E_n(T)$-equivalent to $\left(\bigvee_{i \in s} \phi_i\right)$ for some non-empty subset s of $\{1, .., N\}$.
We assumed that $T \models \exists v_1 \cdots \exists v_n \psi$. If this is not the case, then ψ is $E_n(T)$-equivalent to, say, $\neg v_1 \doteq v_1$. So the number of $E_n(T)$-equivalence classes is at most the number of subsets of $\{1, .., N\}$.
Thus $\left| F_n(\mathcal{L}) / E_n(T) \right| \leq 2^N$ as required.

<u>Case 2</u> There are infinitely many n-types (over T).
Let $p_1, p_2, \ldots, p_r, \ldots$ be the distinct n-types (over T) and (since we are assuming that they are all principal) let $\phi_1, \phi_2, \ldots, \phi_r, \ldots$ be principal formulas for them.

Now if $i \neq j$ then $T \models \neg \exists v_1 \cdots \exists v_n (\phi_i \wedge \phi_j)$ --- (*).
For if $i \neq j$, then $p_i \neq p_j$ so there is some formula $\psi \in p_i$ such that $\psi \notin p_j$, so $\neg\psi \in p_j$. But then $T \models \forall v_1 \cdots \forall v_n (\phi_i \to \psi)$ and $T \models \forall v_1 \cdots \forall v_n (\phi_j \to \neg\psi)$ which implies $T \models \neg \exists v_1 \cdots \exists v_n (\phi_i \wedge \phi_j)$, as required.
(This argument also shows that there are, indeed, at most countably many n-types (over T).)

Now let $q := \{ (\bigwedge_{i \in s} \neg \phi_i) : s \subseteq \{1, 2, \ldots\}, s \text{ finite} \}$

Clearly q is closed under conjunction, and for all $s \subseteq_{\text{fin}} \{1, 2, \ldots\}$, $T \models \exists v_1 \cdots \exists v_n (\bigwedge_{i \in s} \neg \phi_i)$ because

otherwise we would have $T \models \forall v_1 \cdots \forall v_n (\bigvee_{i \in s} \phi_i)$. But if we choose $j_0 \in \{1, 2, \ldots\} \setminus s$, then certainly $T \models \exists v_1 \cdots \exists v_n (\phi_{j_0} \wedge (\bigvee_{i \in s} \phi_i))$, so $T \models \exists v_1 \cdots \exists v_n (\bigvee_{i \in s} (\phi_{j_0} \wedge \phi_i))$

so $T \models \bigvee_{i \in s} \exists v_1 \cdots \exists v_n (\phi_{j_0} \wedge \phi_i)$, which clearly contradicts (*).
This shows that q is a partial n-type (over T), and hence, by 13.3(1), $q \subseteq p$ for some n-type p (over T).
But $p = p_{i_0}$ for some i_0, so $\phi_{i_0} \in p_{i_0}$. But

$\neg \phi_{i_0} \in q \subseteq p_{i_0}$, so $(\phi_{i_0} \wedge \neg \phi_{i_0}) \in p_{i_0}$ - contradiction.

So we have shown that there must be at least one non-principal n-type (over T)

$\square$

13.9 <u>Exercise</u>

Let $T = Th(\langle \mathbb{Z} ; + ; 0, 1\rangle)$. Prove that there are uncountably many 1-types (over T).

13.10 <u>Exercise</u>

Let T be ω-categorical and let $\mathfrak{A} \models T$. For $S \subseteq A$, denote by $\langle S \rangle$ the smallest substructure of $\mathfrak{A}$ containing the set S in its domain (see 3.5).

Prove that if S is finite then so is $\langle S \rangle$.