12.6 <u>Exercise</u>

Let $\mathfrak{A} = \langle \mathbb{N}; <; +, \cdot; 0, 1 \rangle$. Use 12.2 to prove that there exists a structure $\mathfrak{B}$ such that $\mathfrak{B} \equiv \mathfrak{A}$ but $\mathfrak{B} \not\cong \mathfrak{A}$. (Such structures $\mathfrak{B}$ are called <u>non-standard models</u> of <u>number theory</u>, and they were the subject of my Ph.D. thesis.)

13. <u>Types</u>.

To prove the $\Rightarrow$ direction in 12.2 we require some more general theory. Here, $\mathcal{L}$ can be any language and T any complete $\mathcal{L}$-theory.

13.1 <u>Definition</u>

Let $n \geq 0$. A subset $p \subseteq F_n(\mathcal{L})$ is called an <u>n-type (over T)</u> if the following three conditions are satisfied:

(1) if $\phi \in p$, then $T \models \exists v_1 \cdots \exists v_n \phi$;
(2) if $\phi \in p$ and $\psi \in p$ then $(\phi \wedge \psi) \in p$;
(3) for any $\phi \in F_n(\mathcal{L})$, either $\phi \in p$ or $\neg\phi \in p$ (but not both by (2) and (1)).

If p just satisfies (1) and (2) we call p a <u>partial n-type (over T)</u>.

<u>Notation</u>: If $\phi \in F_n(\mathcal{L})$ and $\tau_1, \ldots, \tau_n$ are closed terms of $\mathcal{L}$, we write $\phi(\tau_1, \ldots, \tau_n)$ for the sentence obtained by replacing each free occurrence of v_i in ϕ by τ_i (for each $i = 1, \ldots, n$).

Notice that if p is any partial n-type (over T) and $\phi_1, \ldots, \phi_r \in p$ then $\left(\bigwedge_{i=1}^{r} \phi_i\right) \in p$ (by repeated use of 13.1(2)), and hence $T \models \exists v_1 \cdots \exists v_n \left(\bigwedge_{i=1}^{r} \phi_i \right)$.

13.2 <u>Theorem</u>

If p is a partial n-type (over T), then there exists $\mathfrak{A} \models T$ and $a_1, \ldots, a_n \in A$ such that for all $\phi \in p$, $\mathfrak{A} \models \phi[a_1, \ldots, a_n]$. Further, if $\mathcal{L}$ is countable, then $\mathfrak{A}$ may be taken to be countable.

<u>Proof.</u>

Let $c_1, \ldots, c_n$ be new constant symbols and consider the set

$$T \cup \{ \phi(c_1, \ldots, c_n) : \phi \in p \}.$$

This is finitely satisfiable since if $\phi_1, \ldots, \phi_r \in p$ then by the above remark, any $\mathcal{L}$-structure $\mathfrak{A} \models T$ satisfies $\mathfrak{A} \models \exists v_1 \cdots \exists v_n \left(\bigwedge_{i=1}^{r} \phi_i \right)$. So for some $a_1, \ldots, a_n \in A$ we have

$$\mathfrak{A} \models \left(\bigwedge_{i=1}^{r} \phi_i \right) [a_1, \ldots, a_n]$$

which easily implies

$$\langle \mathfrak{A}, a_1, \ldots, a_n \rangle \models \left(\bigwedge_{i=1}^{r} \phi_i(c_1, \ldots, c_n) \right)$$

Thus, by the compactness theorem there is some $\mathfrak{B} \models T$ and $b_1, \ldots, b_n \in B$ such that for all $\phi \in p$,

$$\langle \mathfrak{B}, b_1, \ldots, b_n \rangle \models \phi(c_1, \ldots, c_n), \text{ so for all } \phi \in p,$$

$$\mathfrak{B} \models T \text{ and } \mathfrak{B} \models \phi[b_1, \ldots, b_n], \text{ as required.}$$

Now if $\mathcal{L}$ is countable we just use the downward Löwenheim-Skolem Theorem to find $\mathfrak{B}' \preccurlyeq \mathfrak{B}$ with $b_1, \ldots, b_n \in B'$ and B' countable. Then $\mathfrak{B}' \models T$ and $\mathfrak{B}' \models \phi[b_1, \ldots, b_n]$ for all $\phi \in p$.

$\square$

13.3 <u>Remarks and definitions</u>

(1) For any $\mathfrak{A} \models T$ and $a_1, \ldots, a_n \in A$, the subset $\{\phi \in F_n(\mathcal{L}) : \mathfrak{A} \models \phi[a_1, \ldots, a_n]\}$ is clearly an n-type (over T). Thus, one consequence of 13.2 is that any partial n-type (over T) may be extended to (i.e. is contained in) some n-type (over T).

(2) We say that an n-type (over T), p say, is <u>realised</u> in a model $\mathfrak{A} \models T$, if there is some $a_1, \ldots, a_n \in A$ such that for all $\phi \in p$, $\mathfrak{A} \models \phi[a_1, \ldots, a_n]$. (and then we say that $\langle a_1, \ldots, a_n \rangle$ <u>realises</u> p in $\mathfrak{A}$). If $\mathfrak{A}$ does not realise p (by any $\langle a_1, \ldots, a_n \rangle \in A^n$) we say that $\mathfrak{A}$ <u>omits</u> p.

(3) Suppose $\mathfrak{A}, \mathfrak{B} \models T$ and $\pi : \mathfrak{A} \cong \mathfrak{B}$. If $\langle a_1, \ldots, a_n \rangle \in A^n$ realises the n-type (over T) p in $\mathfrak{A}$ then, by 6.3, $\langle \pi(a_1), \ldots, \pi(a_n) \rangle$ realises p in $\mathfrak{B}$. Hence, isomorphic models of T realise exactly the same n-types (over T), for all n.

(4) Assuming $\mathcal{L}$ is countable and $n \geq 1$ is such that $F_n(\mathcal{L})/E_n(T)$ is infinite, we shall show

that there is some n-type (over T), p_0 say, and some countable $\mathfrak{A} \models T$ such that $\mathfrak{A}$ omits p_0. Since, by 13.2, there is some countable $\mathfrak{B} \models T$ that realises p_0, it follows from (3) that $\mathfrak{A} \not\cong \mathfrak{B}$, and hence that T is not ω-categorical.

We now discuss where <u>not</u> to look for such a type p_0.