12.6 Exercise

Let \( \mathcal{C} = \langle \mathbb{N}, <, +, \cdot, 0, 1 \rangle \). Use 12.2 to prove that there exists a structure \( \mathcal{D} \) such that
\( \mathcal{D} \cong \mathcal{C} \) but \( \mathcal{D} \not\cong \mathcal{C} \). (Such structures \( \mathcal{D} \) are called \underline{non-standard models of number theory}, and they were the subject of my Ph.D. thesis.)

13. Types.

To prove the \( \Rightarrow \) direction in 12.2 we require some more general theory. Here, \( L \) can be any language and \( T \) any complete \( L \)-theory.

13.1 Definition

Let \( n \geq 0 \). A subset \( p \subseteq F_n(L) \) is called an \underline{\( n \)-type (over \( T \))} if the following three conditions are satisfied:

1. If \( \phi \in p \), then \( T \models \forall x_1 \ldots \forall x_n \phi \);
2. If \( \phi \in p \) and \( \psi \in \neg p \) then \( (\phi \lor \psi) \notin p \);
3. For any \( \phi \in F_n(L) \), either \( \phi \in p \) or \( \neg \phi \in p \) (but not both by (2) and (1)).

If \( p \) just satisfies (1) and (2) we call \( p \) a partial \( \underline{n \text{-}type (over } T) \).

Notation: If \( \phi \in F_n(L) \) and \( \xi_1, \ldots, \xi_n \) are closed terms of \( L \), we write \( \phi(\xi_1, \ldots, \xi_n) \) for the sentence obtained by replacing each free occurrence of \( \xi_i \) in \( \phi \) by \( \xi_i \) (for each \( i = 1, \ldots, n \)).

Notice that if \( p \) is any partial \( n \)-type (over \( T \)) and \( \phi_1, \ldots, \phi_n \in p \) then \( (\bigwedge_{i=1}^n \phi_i) \in p \) (by repeated
use of 13.1(2)), and hence $T = \exists \psi_1 \ldots \exists \psi_n (\bigwedge \phi_i)$.

13.2 Theorem

If $p$ is a partial $n$-type (over $T$), then there exists $C \models T$ and $a_1, \ldots, a_n \in A$ such that for all $\phi \in p$, $C \models \phi[a_1, \ldots, a_n]$. Further, if $L$ is countable, then $C$ may be taken to be countable.

Proof.

Let $c_1, \ldots, c_m$ be new constant symbols and consider the set

$T \cup \{ \phi(c_1, \ldots, c_m) : \phi \in p \}$.

This is finitely satisfiable since if $\phi_1, \ldots, \phi_k \in p$ then by the above remark, any $L$-structure $C \models T$ satisfis $C \models \exists \psi_1 \ldots \exists \psi_n (\bigwedge \phi_i)$. So for some $a_1, a_2, \ldots, a_m \in A$ we have

$C \models (\bigwedge \phi_i)[a_1, \ldots, a_m]$

which easily implies

$\langle C, a_1, \ldots, a_m \rangle \models (\bigwedge \phi_i(c_1, \ldots, c_m))$.

Thus, by the compactness theorem there is some $L' \models T$ and $b_1, \ldots, b_m \in B$ such that for all $\phi \in p$, $\langle L', b_1, \ldots, b_m \rangle \models \phi[c_1, \ldots, c_m]$, so for all $\phi \in p$,

$L' \models T$ and $L' \models \phi[b_1, \ldots, b_m]$, as required.

Now if $L$ is countable we just use the downward Löwenheim-Skolem Theorem to find $L' \leq L$ with $b_1, \ldots, b_m \in B'$ and $B'$ countable. Then $L' \models T$ and $L' \models \phi[b_1, \ldots, b_m]$ for all $\phi \in p$. 

□
13.3 Remarks and definitions

(1) For any \( \mathcal{A} \models T \) and \( a_1, \ldots, a_n \in \mathcal{A} \), the subset
\[ \{ \phi \in \mathcal{F}_n(T) : \mathcal{A} \models \phi[a_1, \ldots, a_n] \} \]
\[ \text{is clearly an \( n \)-type (over \( T \)).} \]
Thus, one consequence of 13.2 is that any partial \( n \)-type (over \( T \)) may be extended to (i.e., is contained in) some \( n \)-type (over \( T \)).

(2) We say that an \( n \)-type (over \( T \)), \( \vdash \), is realized in a model \( \mathcal{A} \models T \), if there is some \( a_1, \ldots, a_n \in \mathcal{A} \) such that for all \( \phi \in \vdash \), \( \mathcal{A} \models \phi[a_1, \ldots, a_n] \).

(3) Suppose \( \mathcal{A}, \mathcal{B} \models T \) and \( \pi : \mathcal{A} \cong \mathcal{B} \). If \( \langle a_1, \ldots, a_n \rangle \in \mathcal{A}^n \) realizes the \( n \)-type (over \( T \)) \( p \) in \( \mathcal{A} \), then, by 6.3, \( \langle \pi(a_1), \ldots, \pi(a_n) \rangle \) realizes \( p \) in \( \mathcal{B} \).

Hence, isomorphic models of \( T \) realize exactly the same \( n \)-types (over \( T \)), for all \( n \).

(4) Assuming \( L \) is countable and \( n \geq 1 \) is such that \( \mathcal{F}_n(L) \) is infinite, we shall show
\[ \mathcal{E}_n(T) \]
that there is some \( n \)-type (over \( T \)), \( \vdash \), and some countable \( \mathcal{A} \models T \) such that \( \mathcal{A} \) omits \( \vdash \). Since, by 13.2, there is some countable \( \mathcal{B} \models T \) that realizes \( \vdash \), it follows from (3) that \( \mathcal{A} \cong \mathcal{B} \), and hence that \( T \) is not \( \omega \)-categorical.

We now discuss where not to look for such a type \( \vdash \).