## 12 A characterization of $\omega$-categorical theories.

We fix a countable language $\mathcal{L}$ throughout this section.

**12.1 Definitions**

Let $n \geq 0$.

(1) $F_n(\mathcal{L})$ denotes the set of all $\mathcal{L}$-formulas $\phi$ all of whose free variables are amongst $v_1, \ldots, v_n$. (Thus $F_0(\mathcal{L})$ is the set of sentences of $\mathcal{L}$.)

(2) If $\phi \in F_n(\mathcal{L})$ and $\bar{a} = \langle a_1, a_2, \ldots, a_n, \ldots \rangle \in A^\omega$, where $\mathfrak{A} = \langle A; \ldots \rangle$ is an $\mathcal{L}$-structure, then we know that whether or not $\mathfrak{A} \models \phi[\bar{a}]$ only depends on $a_1, \ldots, a_n$. So we just write $\mathfrak{A} \models \phi[a_1, \ldots, a_n]$ if $\mathfrak{A} \models \phi[\bar{a}]$ for some (i.e. any) $\bar{a} \in A^\omega$ that begins with $a_1, \ldots, a_n$. We also let

$$\phi^{\mathfrak{A}} := \{ \langle a_1, \ldots, a_n \rangle \in A^n : \mathfrak{A} \models \phi[a_1, \ldots, a_n] \}$$

and call $\phi^{\mathfrak{A}}$ the set defined by $\phi$. Sets of this form are called definable (subsets of $A^n$).

(3) For $T$ an $\mathcal{L}$-theory, $E_n(T)$ denotes the relation on $F_n(\mathcal{L})$ defined by

for $\phi, \psi \in F_n(\mathcal{L})$, $\phi \, E_n(T) \, \psi :\Longleftrightarrow T \models \forall v_1 \cdots \forall v_n (\phi \leftrightarrow \psi)$.

(This is clearly equivalent to:

$\phi \, E_n(T) \, \psi \;:\Longleftrightarrow$ for all $\mathfrak{A} \models T$, $\phi^{\mathfrak{A}} = \psi^{\mathfrak{A}}$.)

Clearly $E_n(T)$ is an equivalence relation on $F_n(\mathcal{L})$ and we write $F_n(\mathcal{L})/E_n(T)$ for the set of equivalence classes.

The purpose of this section is to prove the famous:

12.2 <u>Theorem</u> (Ryll-Nardzewski Theorem)

Let $T$ be a complete theory without finite models. Then $T$ is $\omega$-categorical if and only if for all $n \geq 0$, $F_n(\mathcal{L})/E_n(T)$ is finite.

12.3 <u>Remarks</u>

(1) The condition is clearly equivalent to the following: for all $n \geq 0$ there exists $d$ (depending only on $n$) such that for every $\mathfrak{A} \models T$, there are at most $d$ definable subsets of $A^n$. (For the $\Rightarrow$ direction one needs 12.4 below.)

(2) The assumption that $T$ is complete is equivalent to saying $\left| F_0(\mathcal{L})/E_0(T) \right| = 2$.

<u>Proof of 12.2 in the $\Leftarrow$ direction.</u>

Suppose that $F_n(\mathcal{L})/E_n(T)$ is finite for all $n \geq 0$ --- $(*)$

12.4 <u>Exercise</u>

If $\Sigma$ is <u>any</u> complete $\mathcal{L}$-theory, and $\phi$ is an $\mathcal{L}$-sentence, then if $\mathfrak{A} \models \phi$ for <u>some</u> model $\mathfrak{A} \models T$, then $\mathfrak{A} \models \phi$ for <u>every</u> model $\mathfrak{A} \models T$, and hence $T \models \phi$.

Now, back to $T$ as in $(*)$. We first prove the following

<u>Claim</u>: For any $\mathfrak{A} \models T$, $n \geq 1$ and $a_1, \ldots, a_n \in A$, there exists a formula $\phi \in F_n(\mathcal{L})$ such that (i) $\mathfrak{A} \models \phi[a_1, \ldots, a_n]$ and (ii) if $\psi \in F_n(\mathcal{L})$ also satisfies $\mathfrak{A} \models \psi[a_1, \ldots, a_n]$, then $T \models \forall v_1, \ldots, v_n (\phi \to \psi)$.

<u>Proof of Claim</u>: Let $d = \left| F_n(\mathcal{L})/E_n(T) \right|$ and pick representatives

$\phi_1, \ldots, \phi_d$ of the $E_n(T)$-equivalence classes.
Let $s = \{i : 1 \leq i \leq d$ and $\mathcal{A} \models \phi_i[a_1, \ldots, a_n]\}$ and define
$\phi := \left(\bigwedge_{i \in s} \phi_i\right)$.

Clearly $\mathcal{A} \models \phi[a_1, \ldots, a_n]$. Suppose $\psi \in F_n(\mathcal{L})$ and $\mathcal{A} \models \psi[a_1, \ldots, a_n]$. Let $\phi_i$ be the chosen representative of the $E_n(T)$-equivalence class of $\psi$. Then $T \models \forall v_1 \ldots \forall v_n (\psi \Leftrightarrow \phi_i)$, so since $\mathcal{A} \models T$, $\mathcal{A} \models \forall v_1 \cdots \forall v_n (\psi \Leftrightarrow \phi_i)$. But $\mathcal{A} \models \psi[a_1, \ldots, a_n]$, so $\mathcal{A} \models \phi_i[a_1, \ldots, a_n]$. Hence $i \in s$, so clearly $\mathcal{A} \models \forall v_1 \ldots \forall v_n (\phi \to \phi_i)$. As $\mathcal{A} \models \forall v_1 \ldots \forall v_n (\phi_i \to \psi)$, we have $\mathcal{A} \models \forall v_1 \ldots \forall v_n (\phi \to \psi)$. So $T \models \forall v_1, \ldots, \forall v_n (\phi \to \psi)$ by 12.4.

$\square$ (Claim)

<u>Terminology</u>: A $\phi$ as in the claim is called a <u>principal formula</u> for $a_1, \ldots, a_n$ (in $\mathcal{A}$).

Now to prove the $\omega$-categoricity of $T$ we use a back-and-forth construction. So let $\mathcal{A} \models T$, $\mathcal{B} \models T$ with $A, B$ both countably infinite. Say $A = \{a_1, a_2, \ldots\}$, $B = \{b_1, b_2, \ldots\}$. We construct new enumerations $a_1', a_2', \ldots$ and $b_1', b_2', \ldots$ of $A$ and $B$ respectively such that for all $n \geq 0$ we have

$(*)_n$ for all $\psi \in F_n(\mathcal{L})$, $\mathcal{A} \models \psi[a_1', \ldots, a_n'] \Leftrightarrow \mathcal{B} \models \psi[b_1', \ldots, b_n']$.

Notice that this holds for $n = 0$ by the completeness of $T$ (so $\mathcal{A} \equiv \mathcal{B}$).
Now suppose that we have constructed $a_1', \ldots, a_n'$, $b_1', \ldots, b_n'$ so that $(*)_n$ holds.

<u>Case 1</u> <u>$n$ is odd</u>

Choose $r$ minimal such that $a_r \notin \{a_1', \ldots, a_n'\}$, and set $a_{n+1}' := a_r$.

Let $\phi \in F_{n+1}(\mathcal{L})$ be a principal formula for $a_1', \ldots, a_{n+1}'$ (in $\mathcal{A}$), as given by the Claim.
Let $\phi^* \in F_n(\mathcal{L})$ be the formula $\exists v_{n+1}\, \phi$.
Since $\mathcal{A} \models \phi[a_1', \ldots, a_{n+1}']$ we have $\mathcal{A} \models \phi^*[a_1', \ldots, a_n']$ and hence $\mathcal{B} \models \phi^*[b_1', \ldots, b_n']$ by our inductive assumption $(*)_n$.
So there is some $b \in B$ such that $\mathcal{B} \models \phi[b_1', \ldots, b_n', b]$ and we let $b_{n+1}'$ be such an element $b$.
Then $(*)_{n+1}$ is satisfied because if $\psi \in F_{n+1}(\mathcal{L})$ is such that $\mathcal{A} \models \psi[a_1', \ldots, a_{n+1}']$ then $T \models \forall v_1 \cdots \forall v_{n+1} (\phi \to \psi)$ (because $\phi$ is principal for $a_1', \ldots, a_{n+1}'$ (in $\mathcal{A}$)) and so $\mathcal{B} \models \forall v_1 \cdots \forall v_{n+1} (\phi \to \psi)$. But $\mathcal{B} \models \phi[b_1', \ldots, b_{n+1}']$ (by the choice of $b_{n+1}'$), so $\mathcal{B} \models \psi[b_1', \ldots, b_{n+1}']$ as required. (The $\Leftarrow$ direction in $(*)_{n+1}$ follows by considering $\neg\psi$.)

<u>Case 2</u> <u>$n$ is even</u>

Choose $r$ minimal such that $b_r \notin \{b_1', \ldots, b_n'\}$ and repeat the Case 1 construction (with $b_{n+1}' = b_r$), with the rôles of $\mathcal{A}$ and $\mathcal{B}$ interchanged, to find $a_{n+1}' \in A$ to satisfy $(*)_{n+1}$.

This completes the construction. The fact that the correspondence $a_i' \longmapsto b_i'$ $(i = 1, 2, \ldots)$ defines a bijection from $A$ to $B$ follows just as in the proof of 11.8.3. The fact that this bijection is an isomorphism follows from $(*)_1, (*)_2, (*)_3, \ldots$ (exercise).
Thus $T$ is $\omega$-categorical.

$\square$

12.5 <u>Exercise</u>

Suppose $\mathcal{L}$ is a finite, purely relational language and $T$ is a complete $\mathcal{L}$-theory without finite models. Suppose that for all $n \geq 1$, each $E_n(T)$-equivalence class contains a QF formula. Prove that $T$ is $\omega$-categorical. (In fact, DLO has this property.)

12.6 <u>Exercise</u>

Let $\mathfrak{A} = \langle \mathbb{N} ; < ; +, \cdot ; 0, 1 \rangle$. Use 12.2 to prove that there exists a structure $\mathfrak{B}$ such that $\mathfrak{B} \equiv \mathfrak{A}$ but $\mathfrak{B} \not\cong \mathfrak{A}$. (Such a structure $\mathfrak{B}$ is called a <u>non-standard model of</u> <u>number theory</u>, and they were the subject of my Ph.D. thesis.)