We must show that $\langle A; R\rangle \cong \langle B; S\rangle$ and we use the <u>back-and-forth</u> construction.

So, enumerate A as $\{a_1, a_2, \ldots, a_n, \ldots\}$ and B as $\{b_1, b_2, \ldots, b_n, \ldots\}$. We construct new non-repetitive enumerations $A = \{a_1', a_2', \ldots, a_n', \ldots\}$ and $B = \{b_1', b_2', \ldots, b_n', \ldots\}$ with the property that for all $n \geq 1$,

$(*)_n$ for all $1 \leq i, j \leq n$, $\langle a_i', a_j'\rangle \in R \iff \langle b_i', b_j'\rangle \in S$.

Since the new enumeration will be without repetitions, it follows that the correspondence $a_i' \longmapsto b_i'$ $(i \geq 1)$ defines an isomorphism from $\langle A; R\rangle$ to $\langle B; S\rangle$.

We use induction on n. For $n = 1$ we may take $a_1' = a_1$ and $b_1' = b_1$, since $\langle a_1, a_1\rangle \notin R$ and $\langle b_1, b_1\rangle \notin S$ (by the graph axiom G). Now suppose that $a_1', \ldots, a_n', b_1', \ldots, b_n'$ have been constructed to satisfy $(*)_n$ for some $n \geq 1$.

<u>Case 1</u> <u>n is odd.</u>

Choose d minimal such that $a_d \notin \{a_1', \ldots, a_n'\}$ and put $a_{n+1}' = a_d$.

Let $U := \{i : 1 \leq i \leq n$ and $\langle a_{n+1}', a_i'\rangle \in R\}$ and

$V := \{i : 1 \leq i \leq n$ and $\langle a_{n+1}', a_i'\rangle \notin R\}$.

Since $b_1', \ldots, b_n'$ is non-repetitive, it follows that $\{b_i' : i \in U\} \cap \{b_i' : i \in V\} = \emptyset$ and so, since $\langle B, S\rangle \models \Gamma_n$, it follows easily that there is some $b \in B$ such that $\langle b, b_i'\rangle \in S$ for each $i \in U$, i.e. for each i such that $\langle a_{n+1}', a_i'\rangle \in R$, and $\langle b, b_i'\rangle \notin S$ for each i such that $\langle a_{n+1}', a_i'\rangle \notin R$. So setting b_{n+1}' to be such an element b of B guarantees that $(*)_{n+1}$ holds.

<u>Case 2</u> <u>n is even</u>

Choose d minimal such that $b_d \notin \{b_1', \ldots, b_n'\}$ and repeat the above process with the rôles of $\langle A; R \rangle$ and $\langle B; S \rangle$ interchanged.

This completes the induction. Clearly our new enumerations are non-repetetive. To see that $\{a_n' : n \geq 1\}$ is <u>all</u> of A, suppose it isn't. Choose d minimal such that $a_d \notin \{a_n' : n \geq 1\}$. So $a_1, \ldots, a_{d-1}$ have all been considered by some stage, say $a_1, \ldots, a_{d-1} \in \{a_1', \ldots, a_k'\}$. But then at the next odd stage we will have set a_{k+1}' or a_{k+2}' (depending whether k is odd or even) equal to a_d – a contradiction.

Similarly, $\{b_1', b_2', \ldots\}$ is all of B.

$\square$

11. 8.4 <u>Remark</u>

The unique countable model of Γ_∞ is called the <u>random graph</u>

11.9 <u>Exercise</u>

(1) Prove the 0-1 law for sentences of the language containing just n constant symbols $c_1, \ldots, c_n$.

(2) Write down a sentence in the language with just one binary relation symbol P, E_2 say, with the property that $\langle A; R \rangle \models E_2$ iff R is an equivalence relation on A with just two equivalence classes. Prove that for any sentence ϕ of this language either

$$\frac{\Theta_{E_2}(\phi; N)}{\Theta(E_2; N)} \to 1 \text{ as } N \to \infty \quad \text{or} \quad \frac{\Theta_{E_2}(\phi; N)}{\Theta(E_2; N)} \to 0 \text{ as } N \to \infty.$$

where $\Theta_{E_2}(\phi; N) := \Theta((\phi \wedge E_2); N)$.