Then $\theta(\mathcal{L};N) = N^N$. Let ϕ be the sentence $\forall v_1 \neg F(v_1) \simeq v_1$. Then $\theta(\phi;N) = (N-1)^N$.

So $p(\phi;N) = \dfrac{(N-1)^N}{N^N} = \left(1-\frac{1}{N}\right)^N \to e^{-1}$ as $N\to\infty$.

11.8 <u>The Random Graph.</u>

We now consider the language $\mathcal{L}$ containing just one binary relation symbol P. Rather than proving the 0-1 law for sentences of $\mathcal{L}$ (which is true by 11.6) we consider just <u>graphs</u>: i.e. $\mathcal{L}$-structures in which P is intepreted as a symmetric, irreflexive relation. So if we let G be the sentence

$$(\forall v_1 \forall v_2 (P(v_1,v_2) \to P(v_2,v_1)) \wedge \forall v_1 \neg P(v_1,v_1))$$

then we shall only consider models of G. So our aim is to prove

11.8.1 <u>Theorem</u> (0-1 law for graphs)

For any $\mathcal{L}$-sentence ϕ, either $p_G(\phi;N) \to 1$ as $N\to\infty$ or $p_G(\phi;N)\to 0$ as $N\to\infty$, where

$$p_G(\phi;N) := \frac{\theta_G(\phi;N)}{\theta(G;N)} = \frac{\theta_G(\phi;N)}{2^{\binom{N}{2}}}.$$

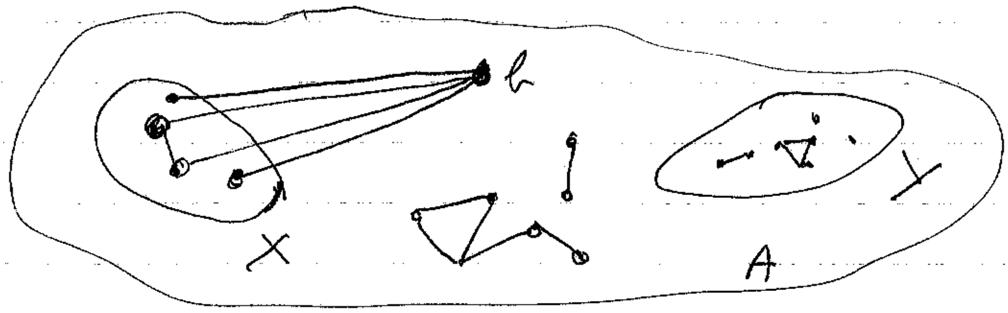
where $\theta_G(\phi;N) := \theta((\phi\wedge G);N)$.

In order to prove 11.8.1, we consider the theory $T^{as}(G) := \{\phi \text{ an } \mathcal{L}\text{-sentence} : p_G(\phi;N)\to 1 \text{ as } N\to\infty\}$. Now by just following the proof of 11.5, Theorem 11.9 will follow if we can show that $T^{as}(G)$ is complete. To do this we consider, for each $n\geq 2$, the sentence

$$\Gamma_n : \forall v_1,\ldots,v_n \forall w_1,\ldots,w_n\left(\left(\bigwedge_{i=1}^{n}\bigwedge_{j=1}^{n}\neg(v_i \simeq w_j)\right)\to \exists u\left(\left(\bigwedge_{i=1}^{n} P(u,v_i)\right)\wedge\left(\bigwedge_{j=1}^{n}\neg P(u,w_j)\right)\right)\right)$$

(where $w_j = v_{n+j}$, $u = v_{2n+1}$ for $j=1,\ldots,n$.)

Thus, $\langle A;R\rangle \models (\Gamma_n \wedge G)$ iff for all $X \subseteq A$, $Y \subseteq A$ with $|X| \leq n$, $|Y| \leq n$ and $X \cap Y = \emptyset$, there exists $b \in A$ such that $\langle b,x\rangle \in R$ for each $x \in X$ and $\langle b,y\rangle \notin R$ for each $y \in Y$:–



We shall establish the following:

11.8.2 for each $n \geq 2$, $\Gamma_n \in T^{as}(G)$, and

11.8.3 $\Gamma_\infty := \{\Gamma_n : n \geq 2\} \cup \{G\}$ is ω-categorical.

Since Γ_∞ has no finite models (easy exercise) it follows from Vaught's test that Γ_∞ is complete, and hence, by 11.8.2, so is $T^{as}(G)$, as required. (The fact that $T^{as}(G)$ is satisfiable follows using the same proof as was used to prove 11.3.)

<u>Proof of 11.8.2</u>

Let $n \geq 2$ be fixed. Let $N \geq 2n+1$ and consider subsets $X, Y \subseteq \{1,\ldots,N\}$ with $X \cap Y = \emptyset$, $|X| = k$, $|Y| = l$ where $1 \leq k \leq n$, $1 \leq l \leq n$. Then the number of graphs with domain $\{1,\ldots,N\}$ that <u>fail</u> to satisfy Γ_n for this particular X, Y is

$$2^{\binom{k}{2}} \cdot 2^{\binom{l}{2}} \cdot 2^{\binom{N-(k+l)}{2}} \cdot \left(2^{k+l}-1\right)^{N-(k+l)}$$

(since such a graph is arbitrary on $\{1,\ldots,N\} \setminus (X \cup Y)$, X, and Y, for each $b \in \{1,\ldots,N\} \setminus \{X \cup Y\}$, edges from b to $X \cup Y$ can R-relate to any subset of $X \cup Y$ except for the one possibility that would satisfy Γ_n, namely $\langle b,x\rangle \in R$ iff $x \in X$).

So the number of graphs with domain $\{1,\dots,N\}$ such that they fail to satisfy Γ_n for some such X and Y is at most

$$\sum_{k=1}^{n}\sum_{\ell=1}^{n} 2^{\binom{k}{2}+\binom{\ell}{2}}\binom{N}{k}\cdot\binom{N-k}{\ell}\cdot 2^{\binom{N-(k+\ell)}{2}}\cdot\left(2^{k+\ell}-1\right)^{N-(k+\ell)} \quad \cdots (*)$$

Now $\binom{N}{k}\binom{N-k}{\ell} \leq N^{2n}$, $\quad 2^{\binom{k}{2}+\binom{\ell}{2}} \leq 2^{n^2}$,

$\binom{N-(k+\ell)}{2} \leq \binom{N}{2} - N(k+\ell)$, and

$$\left(2^{k+\ell}-1\right)^{N-(k+\ell)} \leq 2^{N(k+\ell)}\left(1-\frac{1}{2^{k+\ell}}\right)^N \leq 2^{N(k+\ell)}\cdot\left(1-\frac{1}{2^{2n}}\right)^N$$

for $1 \leq k,\ell \leq n$ and $N \geq 2n+1$.

Hence $(*)$ is bounded by

$$2^{n^2}\cdot\sum_{k=1}^{n}\sum_{\ell=1}^{n} N^{2n}\cdot 2^{\binom{N}{2}-N(k+\ell)}\cdot 2^{N(k+\ell)}\cdot\left(1-\frac{1}{2^{2n}}\right)^N$$

$$= 2^{n^2}\cdot n^2\cdot N^{2n}\cdot 2^{\binom{N}{2}}\cdot\left(1-\frac{1}{2^{2n}}\right)^N$$

So $$\frac{\theta(\neg\Gamma_n;N)}{\theta(G;N)} \leq 2^{n^2}\cdot n^2\cdot N^{2n}\cdot\left(1-\frac{1}{2^{2n}}\right)^N \longrightarrow 0 \text{ as } N\to\infty.$$

$$\therefore \frac{\theta(\Gamma_n;N)}{\theta(G;N)} = \frac{\theta(G;N)-\theta(\neg\Gamma_n;N)}{\theta(G;N)} \to 1 \text{ as } N\to\infty.$$

$\square$

<u>Proof of 11.8.3</u>

Let $\langle A;R\rangle$, $\langle B;S\rangle$ be two countably infinite models of Γ_∞. (In particular, they are both graphs.)