The 0-1 law for relational structures.

We fix a finite language \( L \) (i.e. \( L = L_0 \) where \( \sigma = \langle I, J, K, \rho, \pi \rangle \) and \( I, J, K \) are finite sets.)

Let \( S(L; N) \) be the set of all \( L \)-structures with domain \( \{1, \ldots, N \} \), and let \( \Theta(L; N) = |S(L; N)| \) be the number of them. This is clearly finite.

11.1 Exercise
Write down a formula for \( \Theta(L; N) \) in terms of \( I, J, K, \rho, \pi \).

If \( \phi \) is a sentence of \( L \) we let \( S(\phi; N) \) denote the set of \( C \in S(L; N) \) such that \( C \models \phi \), and define
\[
\Theta(\phi; N) := |S(\phi; N)|.
\]

11.2 Definition
(1) \( p(\phi; N) \) denotes the proportion of \( L \)-structures with domain \( \{1, \ldots, N \} \) that satisfy the \( L \)-sentence \( \phi \):
\[
p(\phi; N) := \frac{\Theta(\phi; N)}{\Theta(L; N)}.
\]

(2) We say that the \( L \)-sentence \( \phi \) is true almost surely if \( p(\phi; N) \to 1 \) as \( N \to \infty \).

(3) \( T_{\text{as}}(L) := \{ \phi \text{ an } L \text{-sentence : } \phi \text{ is true almost surely} \} \).

11.3 Theorem
\( T_{\text{as}}(L) \) is an \( L \)-theory (i.e. it is satisfiable) with no finite models.
Proof. We use the Compactness Theorem to show $T^o(L)$ is satisfiable. So let $E_0$ be a finite subset of $T^o(L)$. Say $E_0 = \{ \phi_1, \ldots, \phi_r \}$. Let $\varepsilon = \frac{1}{r+1}$. Then for each $i = 1, \ldots, r$, there exists $N_i$ s.t. $|P(\phi_i; N)| < \varepsilon$ for all $N \geq N_i$. So choose any $N > \max\{M, N_1, \ldots, N_r\}$. We then have

$$0 \leq \frac{\theta(L; N) - \theta(\phi_i; N)}{\theta(L; N)} < \frac{1}{r+1}$$

for all $i = 1, \ldots, r$.

But $\theta(L; N) - \theta(\phi_i; N) = \theta(-\phi_i; N)$, so

$$\theta(-\phi_i; N) < \frac{\theta(L; N)}{r+1}$$

for all $i = 1, \ldots, r$.

So

$$\bigcup_{i=1}^{r} S(-\phi_i; N) \subseteq \frac{r}{r+1} \theta(L; N) < \theta(L; N).$$

So there exists at least one $L$-structure in $S(L; N)$, \textit{i.e.}, not in $\bigcup_{i=1}^{r} S(-\phi_i; N)$. Clearly $G \models \phi_i$ for $i = 1, \ldots, r$.

\textit{i.e.}, $G \models E_0$. So $T^o(L)$ is f.s. and hence satisfiable.

Further, for each fixed $n$, $p(X_n; N) = 1$ for $N \geq n$, (see 9.13), so certainly $X_n \in T^o(L)$. Thus $T^o(L)$ has no finite models.

\[ \square \]

11.4 Exercise

Prove that for any $\phi_1, \ldots, \phi_r \in T^o(L)$, $(\bigwedge_{i=1}^{r} \phi_i) \in T^o(L)$.

11.5 Corollary

Suppose that our language $L$ is such that $T^o(L)$ is a complete theory. Then the 0-1 law
holds for $L$-sentences, i.e. for any $L$-sentence $\phi$ we have that either $p(\phi; N) \to 1$ as $N \to \infty$ or else $p(\phi; N) \to 0$ as $N \to \infty$.

Proof

Let $\phi$ be any $L$-sentence. If $T^a(\langle \alpha \rangle) \models \phi$, then by 10.6 there exists a finite subset $\Sigma_0 \subseteq T^a(\langle \alpha \rangle)$ such that $\Sigma_0 \models \phi$. If $\Sigma_0 = \{ \phi_1, \ldots, \phi_n \}$ let

$$\Phi := (\lor_i \phi_i).$$

Then $\Phi \in T^a(\langle \alpha \rangle)$ (by 11.4), and clearly

$$\Phi \models \phi.$$ This last assertion obviously implies that $p(\Phi; N) \leq p(\phi; N) \leq 1$ for all $N$, and hence

$$p(\phi; N) \to 1$$

as $N \to \infty$, by the Sandwich Rule.

Now if not $T^a(\langle \alpha \rangle) \models \phi$, then $T^a(\langle \alpha \rangle) \models \neg \phi$ by the completeness of $T^a(\langle \alpha \rangle)$. By the above $p(\neg \phi; N) \to 1$ as $N \to \infty$, so clearly $p(\phi; N) = 1 - p(\neg \phi; N) \to 1 - 1 = 0$ as $N \to \infty$.

11.6 Proposition (Jagad, 1976)

If $L$ is a purely relational (finite) language (i.e. there are no function or constant symbols), then $T^a(\langle \alpha \rangle)$ is a complete theory and hence the 0-1 law holds. In fact $T^a(\langle \alpha \rangle)$ is $\omega$-categorical (so its completeness follows from 11.3 and Vaught's Test (9.12)).

11.7 Example

The 0-1 law fails if $L$ is allowed to contain function symbols. E.g. take $L$ to be the language with just one unary function symbol $F$. 