11 The 0-1 law for relational structures.

We fix a finite language $\mathcal{L}$. (I.e. $\mathcal{L} = \mathcal{L}_\sigma$ where $\sigma = \langle I, J, K, \rho, \mu \rangle$ and I, J, K are finite sets.)

Let $S(\mathcal{L}; N)$ be the set of all $\mathcal{L}$-structures with domain $\{1, \ldots, N\}$, and let $\Theta(\mathcal{L}; N) := |S(\mathcal{L}; N)|$ be the number of them. This is clearly finite.

11.1 Exercise

Write down a formula for $\Theta(\mathcal{L}; N)$ in terms of I, J, K, ρ, μ.

If ϕ is a sentence of $\mathcal{L}$ we let $S(\phi; N)$ denote the set of $\mathfrak{A} \in S(\mathcal{L}; N)$ such that $\mathfrak{A} \models \phi$, and define $\Theta(\phi; N) := |S(\phi; N)|$.

11.2 Definition

(1) $p(\phi; N)$ denotes the proportion of $\mathcal{L}$-structures with domain $\{1, \ldots, N\}$ that satisfy the $\mathcal{L}$-sentence ϕ:–

$$p(\phi; N) := \frac{\Theta(\phi; N)}{\Theta(\mathcal{L}; N)}.$$

(2) We say that the $\mathcal{L}$-sentence ϕ is true almost surely if $p(\phi; N) \to 1$ as $N \to \infty$.

(3) $T^{as}(\mathcal{L}) := \{\phi \text{ an } \mathcal{L}\text{-sentence} : \phi \text{ is true almost surely}\}$.

11.3 Theorem

$T^{as}(\mathcal{L})$ is an $\mathcal{L}$-theory (i.e. it is satisfiable) with no finite models.

Proof.

We use the Compactness Theorem to show $T^{as}(\mathcal{L})$ is satisfiable. So let Σ_0 be a finite subset of $T^{as}(\mathcal{L})$. Say $\Sigma_0 = \{\phi_1, \ldots, \phi_r\}$. Let $\varepsilon = \frac{1}{r+1}$. Then for each $i = 1, \ldots, r$, there exists N_i s.th. $|1 - p(\phi_i; N)| < \varepsilon$ for all $N \geq N_i$. So choose any $N > \max\{N_1, \ldots, N_r\}$. We then have $0 \leq \dfrac{\theta(\mathcal{L}; N) - \theta(\phi_i; N)}{\theta(\mathcal{L}; N)} < \dfrac{1}{r+1}$ for all $i = 1, \ldots, r$.

But $\theta(\mathcal{L}; N) - \theta(\phi_i; N) = \theta(\neg\phi_i; N)$, so

$$\theta(\neg\phi_i; N) < \frac{\theta(\mathcal{L}; N)}{r+1} \quad \text{for all } i = 1, \ldots, r.$$

So $$\left|\bigcup_{i=1}^{r} \mathcal{S}(\neg\phi_i; N)\right| < \frac{r}{r+1} \cdot \theta(\mathcal{L}; N) < \theta(\mathcal{L}; N).$$

So there exists at least one $\mathcal{L}$-structure in $\mathcal{S}(\mathcal{L}; N)$, $\mathfrak{A}$ say, not in $\bigcup_{i=1}^{r} \mathcal{S}(\neg\phi_i; N)$. Clearly $\mathfrak{A} \models \phi_i$ for $i = 1, \ldots, r$,

i.e. $\mathfrak{A} \models \Sigma_0$. So $T^{as}(\mathcal{L})$ is f.s. and hence satisfiable.

Further, for each fixed n, $p(\chi_n; N) = 1$ for $N \geq n$, (see 9.13), so certainly $\chi_n \in T^{as}(\mathcal{L})$. Thus $T^{as}(\mathcal{L})$ has no finite models.

□

11.4 Exercise

Prove that for any $\phi_1, \ldots, \phi_r \in T^{as}(\mathcal{L})$, $\left(\bigwedge_{i=1}^{r} \phi_i\right) \in T^{as}(\mathcal{L})$.

11.5 Corollary

Suppose that our language $\mathcal{L}$ is such that $T^{as}(\mathcal{L})$ is a complete theory. Then the 0-1 law holds for $\mathcal{L}$-sentences, i.e. for any $\mathcal{L}$-sentence ϕ we have that either $p(\phi; N) \to 1$ as $N \to \infty$ or else $p(\phi; N) \to 0$ as $N \to \infty$.

Proof.

Let ϕ be any $\mathcal{L}$-sentence. If $T^{as}(\mathcal{L}) \models \phi$, then by 10.6 there exists a finite subset $\Sigma_0 \subseteq T^{as}(\mathcal{L})$ such that $\Sigma_0 \models \phi$. If $\Sigma_0 = \{\phi_1, .., \phi_r\}$ let $\Psi := \left(\bigwedge_{i=1}^{r} \phi_i\right)$. Then $\Psi \in T^{as}(\mathcal{L})$ (by 11.4), and clearly

$\Psi \models \phi$. This last assertion obviously implies that $p(\Psi; N) \leq p(\phi; N) \leq 1$ for all N, and hence

$p(\phi; N) \to 1$ as $N \to \infty$, by the Sandwich Rule.

Now if not $T^{as}(\mathcal{L}) \models \phi$, then $T^{as}(\mathcal{L}) \models \neg\phi$ by the completeness of $T^{as}(\mathcal{L})$. By the above $p(\neg\phi; N) \to 1$ as $N \to \infty$, so clearly $p(\phi; N) = 1 - p(\neg\phi; N) \to 1 - 1 = 0$ as $N \to \infty$.

$\square$

11.6 Proposition (Fagin, 1976)

If $\mathcal{L}$ is a purely relational (finite) language (i.e. there are no function or constant symbols), then $T^{as}(\mathcal{L})$ is a complete theory and hence the 0-1 law holds. In fact $T^{as}(\mathcal{L})$ is ω-categorical (so its completeness follows from 11.3 and Vaught's Test (9.12)).

$\square$

11.7 Example

The 0-1 law fails if $\mathcal{L}$ is allowed to contain function symbols. E.g. take $\mathcal{L}$ to be the language with just one unary function symbol F.