actually calculating the integral of a particular function, however, we usually have to resort to the techniques of Riemann integration (and then, possibly, take limits as permitted by the Dominated Convergence Theorem). So we should show that if a function $f:[0,1]\to\mathbb{R}$ is Riemann integrable then it is integrable in our new sense - which we call Lebesgue integrable.

Theorem 3.28*

Let $f:[0,1]\to\mathbb{R}$ be Riemann integrable. Then f is Lebesgue integrable and $\int f\,d\mu = \int_0^1 f(x)\,dx$.

First a lemma which is a standard fact about Riemann integration (though possibly not proved in year 2).

Theorem 3.27

Let $f:[0,1]\to\mathbb{R}$ be (bounded and) Riemann integrable. Then $\int_0^1 f(x)\,dx$ may be calculated by use of the partitions $\Delta_k := \{ i/2^k : i=0,\dots,2^k\}$. I.e. if we set

$m_i^{(k)} := \inf\{ f(x) : \frac{i}{2^k}\le x\le \frac{i+1}{2^k}\}$ and $M_i^{(k)} := \sup\{f(x) : i/2^k \le x\le \frac{i+1}{2^k}\}$

and $L(f,\Delta_k) := \sum_{i=0}^{2^k-1} m_i^{(k)}/2^k$ and $U(f,\Delta_k) := \sum_{i=0}^{2^k-1} M_i^{(k)}/2^k$, then

$$\lim_{k\to\infty} L(f,\Delta_k) = \lim_{k\to\infty} U(f,\Delta_k) = \int_0^1 f(x)\,dx. \qquad \square$$

Proof of 3.28

Let Δ_k be the partition as above. Define simple functions f_k, g_k by

$$f_k = \sum_{i=0}^{2^k-1} m_i^{(k)} \chi_{[i/2^k,\frac{i+1}{2^k})} + f(1)\cdot\chi_{\{1\}}, \quad g_k = \sum_{i=0}^{2^k-1} M_i^{(k)}\chi_{[i/2^k,\frac{i+1}{2^k})} + f(1)\chi_{\{1\}}$$

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* The numbering is as in the printed notes.

where the $m_i^{(k)}, M_i^{(k)}$ are also as above.

Easy to check that $\forall x \in [0,1]$, $\forall k \geq 1$, $f_k(x) \leq f_{k+1}(x)$, $g_k(x) \geq g_{k+1}(x)$
and for all $k, k' \geq 1$, $f_k(x) \leq f(x) \leq g_{k'}(x)$.

Now $\int f_k \, d\mu = \sum_{i=0}^{2^k-1} m_i^{(k)} \mu([\tfrac{i}{2^k}, \tfrac{i+1}{2^k})) = \sum_{i=0}^{2^k-1} \frac{m_i^{(k)}}{2^k} = L(f, \Delta_k)$

and similarly $\int g_k \, d\mu = U(f, \Delta_k)$.

By 3.27 (as f is Riemann integrable) we obtain

$$\left.\begin{aligned} \lim_{k\to\infty} \int f_k \, d\mu &= \lim_{k\to\infty} L(f,\Delta_k) = \int_0^1 f(x)\,dx, \text{ and} \\ \lim_{k\to\infty} \int g_k \, d\mu &= \lim_{k\to\infty} U(f,\Delta_k) = \int_0^1 f(x)\,dx \end{aligned}\right\} (*)$$

Now $\forall x \in [0,1]$, $(f_k(x))_{k\geq 1}$ is an increasing sequence, bounded above by $f(x)$. Let $F(x) = \lim_{k\to\infty} f_k(x)$. By 3.13 F is measurable. Also, $f_1(x) \leq F(x) \leq g_1(x)$ $(\forall x \in [0,1])$, so F is integrable.

Similarly $G := \lim_{k\to\infty} g_k$ is integrable.

Clearly, for all k, $f_k \leq F \leq f \leq G \leq g_k$, so since f_k, F, G, g_k are integrable we have

$$\int f_k \, d\mu \leq \int F \, d\mu \leq \int G \, d\mu \leq \int g_k \, d\mu$$

(by 3.21 (iv)). (We don't know that f is measurable yet.)

Letting $k \to \infty$ and using $(*)$ we get

$$\int_0^1 f(x)\,dx \leq \int F \, d\mu \leq \int G \, d\mu \leq \int_0^1 f(x)\,dx$$

and so they are all equal.

$\therefore \int (G-F)\, d\mu = 0$ (by 3.20), and hence by 3.21 (iii) (since $(G-F)(x) \geq 0$ $\forall x \in [0,1]$) we have that $G - F = 0$ μ-a.e. But $F(x) \leq f(x) \leq G(x)$ $(\forall x \in [0,1])$

so $f(x) = F(x)$ μ-a.e. Hence, by 3.21 (i), f is (Lebesgue-) integrable and $\int f\,d\mu = \int F\,d\mu = \int_0^1 f(x)\,dx$. $\square$

Example (See sheet 5, Q4.)

Let $p > 0$, $p \in \mathbb{R}$. Define $g : [0,1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & \text{if } x = 0; \\ (n+1)^p & \text{if } n \text{ is the unique pos. integer with } \frac{1}{n+1} < x \leq \frac{1}{n} \\ & \text{if } x > 0. \end{cases}$$

Now let (for $N \geq 1$):

$$g_N = \sum_{n=1}^{N} (n+1)^p \chi_{(\frac{1}{n+1}, 1/n]}.$$

Then $(g_N)_{N\geq 1}$ is an increasing sequence of non-negative simple (hence measurable) functions and $g_N \to g$ pointwise as $N \to \infty$. Then g is measurable (by 3.13 (ii)) and, by definition of $\int g\,d\mu$, we have

$$\int g\,d\mu = \lim_{N\to\infty} \int g_N\,d\mu.$$

(Note that $\int g\,d\mu$ is well defined $\in \mathbb{R}^*$ as g is non-negative.)

Now $\int g_N\,d\mu = \sum_{n=1}^{N} (n+1)^p \mu\left((\frac{1}{n+1}, \frac{1}{n}]\right) = \sum_{n=1}^{N} (n+1)^p \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$$= \sum_{n=1}^{N} \frac{(n+1)^p}{n(n+1)} = \sum_{n=1}^{N} \frac{(n+1)^{p-1}}{n}.$$

Now, if $p < 1$, $\sum_{n=1}^{N} \frac{(n+1)^{p-1}}{n} = \sum_{n=1}^{N} \frac{1}{n\cdot(n+1)^{1-p}} \leq \sum_{n=1}^{N} \frac{1}{n^{2-p}}$.

But as $2-p > 1$, $\sum_{n=1}^{N} \frac{1}{n^{2-p}}$ converges, so $\left(\int g_N\,d\mu\right)_{N\geq 1}$ is an increasing, bounded sequence, and hence converges to a finite limit. So $\int g\,d\mu < \infty$ and g is integrable.

On the other hand, if $p \geq 1$, then $\sum_{n=1}^{N} \frac{(n+1)^{p-1}}{n} \geq \sum_{n=1}^{N} \frac{n^{p-1}}{n}$

$= \sum_{n=1}^{N} \frac{1}{n^{2-p}} \geq \sum_{n=1}^{N} \frac{1}{n} \to \infty$ as $N \to \infty$. Hence $\lim_{N\to\infty} \int g_N\,d\mu = \infty$, so $\int g\, d\mu = \infty$, and g is not integrable.

Now let $0 < p < 1$, and consider the function

$$f(x) = \begin{cases} x^{-p} & \text{if } 0 < x \le 1 \\ \infty & \text{if } x = 0 \end{cases}.$$

Let $$f_n(x) = \begin{cases} x^{-p} & \text{if } \frac{1}{n} < x \le 1 \\ n^p & \text{if } 0 \le x \le \frac{1}{n} \end{cases}, \quad (\text{for each } n \ge 1).$$

Then each f_n is continuous (and bounded) on $[0,1]$ and hence Riemann integrable. By 3.28 we have

$$\int f_n\, d\mu = \int_0^1 f_n(x)\, dx = \int_0^{1/n} n^p\, dx + \int_{1/n}^1 x^{-p}\, dx$$

$$= \left[n^p x \right]_0^{1/n} + \left[\frac{x^{-p+1}}{-p+1} \right]_{1/n}^1$$

$$= n^{p-1} + \frac{1}{-p+1}\left[1 - \frac{1}{n^{-p+1}}\right]$$

$$\to \frac{1}{-p+1} \text{ as } n \to \infty \ (\text{since } p-1 < 0).$$

Also, $f_n \to f$ pointwise as $n \to \infty$.

However, apart from $x = 0$, we have $|f_n(x)| \le g(x)$ (for $x \in (0,1]$).

(since if $\frac{1}{k+1} < x \le \frac{1}{k}$, then $g(x) = (k+1)^p$, whereas $f_n(x) = n^p$ if $\frac{1}{k} \le \frac{1}{n}$ (so $n^p \le k^p \le (k+1)^p$), and $f_n(x) = x^{-p}$ if $\frac{1}{k} > \frac{1}{n}$, so

$x^{-p} = \frac{1}{x^p} < (k+1)^p = g(x)$).

Hence, by the Dominated Convergence Theorem (since g is integrable), we have $\int f\, d\mu = \frac{1}{-p+1}$, but f is not Riemann integrable.

(I'll leave you to show that f is not integrable if $p \ge 1$.)

<u>Another example</u> (See 2012 exam, question B10.)

For $x \in [0,1]$ define $f(x) = \sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{k^{3/2}}$. Show that f is a well-defined measurable function and that $\int f\, d\mu = 0$.

Solution

We would like to write

$$\int f(x)\,d\mu = \int \sum_{n=1}^{\infty} \frac{\sin(2\pi kx)}{k^{3/2}}\,d\mu \overset{①}{=} \sum_{n=1}^{\infty} \int \frac{\sin(2\pi kx)}{k^{3/2}}\,d\mu$$

$$\overset{②}{=} \sum_{n=1}^{\infty} \int_0^1 \frac{\sin(2\pi kx)}{k^{3/2}}\,dx$$

$$\overset{③}{=} \sum_{n=1}^{\infty} 0 = 0$$

Step ② follows from 3.28 (since $\frac{\sin(2\pi kx)}{k^{3/2}}$ is continuous on $[0,1]$, hence Riemann integrable) and ③ is by calculation of the Riemann integral. So ① is the issue, and requires the Dominated Convergence Theorem.

Firstly note that $\left|\frac{\sin(2\pi kx)}{k^{1/2}}\right| \le \frac{1}{k^{3/2}}$ and $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges, so $\sum_{k=1}^{\infty} \frac{\sin(2\pi kx)}{k^{1/2}}$ is (absolutely) convergent by the Comparison Test, and hence $f(x)$ is well defined.

Let $f_n(x) = \sum_{k=1}^{n} \frac{\sin(2\pi kx)}{k^{3/2}}$, so that $f_n \to f$ pointwise on $[0,1]$.

Also $|f_n(x)| \le \sum_{k=1}^{n} \left|\frac{\sin(2\pi kx)}{k^{3/2}}\right| \le \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} = A$, say $(A<\infty)$.

Also each f_n is measurable (being continuous on $[0,1]$). So we may apply DCT, with $g(x) =$ constant function A, to conclude that f is integrable and $\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$

$$= \lim_{n\to\infty} \int \sum_{k=1}^{n} \frac{\sin(2\pi kx)}{k^{3/2}}\,d\mu$$

$$= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \int \frac{\sin(2\pi kx)}{k^{3/2}}\,d\mu \right) \quad \text{(by 3.20)}$$

$$= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \int_0^1 \frac{\sin(2\pi kx)}{k^{3/2}}\,dx \right) \quad \text{(by 3.28)}$$

$$= \lim_{n\to\infty} \left(\sum_{k=1}^{n} 0 \right) = \lim_{n\to\infty} 0 = 0, \text{ as required}$$

— You can now attempt sheet 6.

Chapter 4: Fourier Series and $L^2([-\pi,\pi],\mu)$.

Metric spaces

Definition

(a) Let X be a non-empty set. Then a function $d : X \times X \to \mathbb{R}$ is called a metric (on X) if for all $x, y, z \in X$

m(1) $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$;

m(2) $d(x,y) = d(y,x)$;

m(3) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

(b) A sequence $(x_n)_{n\geq 1}$ with $x_n \in X$ $(\forall n \geq 1)$ is called a Cauchy sequence in the metric space (X,d) if $\forall \varepsilon > 0$, $\exists N \geq 1$ such that for all $n, m \geq N$, $d(x_n, x_m) < \varepsilon$.

(c) A sequence $(x_n)_{n\geq 1}$ with $x_n \in X$ $(\forall n \geq 1)$ is said to converge to a point $x \in X$, if $\forall \varepsilon > 0$ $\exists N \geq 1$ such that for all $n \in N$ $d(x_n, x) < \varepsilon$.

(d) A metric space (X,d) is called complete if every Cauchy sequence in (X,d) converges to a point in X.

Inner products on vector spaces

Definition

Let V be a vector space over $\mathbb{R}$. A function $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$ is called a inner product (on V) if for all $u, v, w \in V$ and $a, b \in \mathbb{R}$

p(1) $\langle u, v \rangle = \langle v, u \rangle$;

p(2) $\langle au + bv, w \rangle = a\langle u, v \rangle + b\langle v, w \rangle$;

p(3) $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ if and only if $u = 0$

Example For $\bar{x}, \bar{y} \in \mathbb{R}^n$, $\langle \bar{x}, \bar{y} \rangle := \sum_{i=1}^{n} x_i \cdot y_i$ (where $\bar{x} = (x_1, \ldots, x_n)$, $\bar{y} = (y_1, \ldots, y_n)$) is an inner product on the vector space $\mathbb{R}^n$.

Given an inner product $\langle \cdot , \cdot \rangle$ on a vector space we define the norm $\| \cdot \| : V \to \mathbb{R}$ by $\|v\| := +\langle v, v \rangle^{1/2}$. Then we have for all $v, w \in V$ and $a \in \mathbb{R}$:-

(A) $\|v\| \geq 0$ with $\|v\| = 0$ if and only if $v = 0$;

(B) $\|av\| = |a| \cdot \|v\|$;

(C) $\|v + w\| \leq \|v\| + \|w\|$.

<u>Proof</u>

(A) follows easily from $p(3)$.

For (B), $\|av\|^2 = \langle av, av \rangle = a \langle v, av \rangle$ (by $p(2)$ with $b=0$)

$= a \cdot \langle av, v \rangle$ (by $p(1)$)

$= a^2 \cdot \langle v, v \rangle$ (by $p(2)$ with $b=0$)

Taking positive square roots gives $\|av\| = |a| \cdot \langle v, v \rangle^{1/2} = |a| \cdot \|v\|$.

(C) Here's an amusing proof which I'll leave you to complete using your knowledge of quadratic equations!

Let $x \in \mathbb{R}$ be arbitrary.

By $p(3)$, $\langle xv - u, xv - u \rangle \geq 0$. Now expand the left hand side using $p(2)$ and $p(3)$ ... and obtain an inequality involving $\langle v, v \rangle, \langle u, u \rangle, \langle v, u \rangle$. Plug this into the expansion of $\langle v+u, v+u \rangle$.

<u>Definition</u>

For $\langle \cdot, \cdot \rangle$ an inner product on a vector space V, with associated norm $\|\cdot\|$ (as above), we define the associated <u>metric</u> $d : V \times V \to \mathbb{R}$ by $d(v, w) := \|v - w\|$. {It is an easy exercise using (A), (B), (C) that this is a metric on V.

<u>Example (continued)</u>

$d(\bar{x}, \bar{y})$ (for $\bar{x}, \bar{y} \in \mathbb{R}^n$) is given by $\|\bar{x} - \bar{y}\| = \langle \bar{x} - \bar{y}, \bar{x} - \bar{y} \rangle^{1/2}$

$= \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$, which is the usual Euclidean distance on $\mathbb{R}^n$.

<u>Definition</u>

A vector space V with an inner product $\langle \cdot, \cdot \rangle$ is called a <u>Hilbert space</u>, if it is complete as a metric space (for the associated metric).

(E.g. $\mathbb{R}^n$ in example above is complete.)

Spaces of measurable functions

Definition

Let $f : [-\pi, \pi] \to \mathbb{R}$ be measurable. We say that f is square integrable if $|f|^2$ is integrable, i.e. if

$$\int_{-\pi}^{\pi} |f|^2 \, d\mu < \infty .$$

Then we write $\quad \|f\|_2 := \left(\frac{1}{\pi} \int_{-\pi}^{\pi} |f|^2 d\mu \right)^{1/2}$

and call this the L^2-norm of f ("L-two norm").

Unfortunately, it's not a norm because if $f : [-\pi,\pi] \to \mathbb{R}$ is any function vanishing outside some $E \subseteq [-\pi, \pi]$ with $\mu(E) = 0$, then $\|f\|_2 = 0$ and we could certainly have $f \neq 0$.

So we must (at least) identify two functions if they agree μ-a.e.:—

Define $f \sim g$ iff $f = g$ μ-a.e., and let

$[f]$ = equivalence class of f = $\{ g : [-\pi,\pi] \to \mathbb{R} : f = g \ \mu\text{-a.e.} \}$.

Definition

We write $L^2([-\pi,\pi], \mu, \mathbb{R})$ for the set of equivalence classes of square integrable functions on $[-\pi, \pi]$, i.e.

$$L^2([-\pi,\pi], \mu, \mathbb{R}) := \{ [f] : f \text{ square integrable} \}.$$

Having made the point about equivalence classes we shall, from now on, regard the elements of $L^2([-\pi,\pi], \mu, \mathbb{R})$ as functions. But of course we must observe that all our definitions are invariant under changing the functions on a set of measure 0. Note that this is clear for $\|f\|_2$.

Aim: Let $\langle f, g \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f g \, d\mu$. We aim to show that this is a well-defined inner product on the vector space $L^2([-\pi,\pi], \mu, \mathbb{R})$

~~and~~ with respect to which $L^2([-\pi,\pi],\mu,\mathbb{R})$ is a Hilbert space.

We need some technical lemmas:

Lemma 4.1

For $f, g \in L^2([-\pi,\pi],\mu,\mathbb{R})$

$$\frac{2}{\pi}\int_{-\pi}^{\pi} fg\,d\mu \leq \|f\|_2^2 + \|g\|_2^2$$

with equality if and only if $f = g$ μ-a.e. (i.e. if f and g are the same element of $L^2([-\pi,\pi],\mu,\mathbb{R})$).

Proof.

We have, $0 \leq (f(x)-g(x))^2 = f(x)^2 - 2f(x)\cdot g(x) + g(x)^2$

so $2f(x)\cdot g(x) \leq f(x)^2 + g(x)^2$ $(*)$.

If we ~~do~~ the same for $|f|$ and $|g|$, we see that $|fg| \leq \frac{f^2+g^2}{2}$, so since f^2, g^2 are integrable so is fg.

Hence we may integrate both sides of $(*)$ and divide by π to get the first result.

Now if we have equality, then writing out the integrals on the right we see that $\int_{-\pi}^{\pi}(f-g)^2\,d\mu = 0$. By 3.21, this implies $(f-g)^2 = 0$ μ-a.e., so $f = g$ μ-a.e. □

Now obviously $f \in L^2([-\pi,\pi],\mu,\mathbb{R})$ ~~implies~~ implies $|f| \in L^2([-\pi,\pi],\mu,\mathbb{R})$ so we get

Corollary.

For $f, g \in L^2([-\pi,\pi],\mu,\mathbb{R})$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|fg|\,d\mu \leq \|f\|_2^2 + \|g\|_2^2$$

with equality if and only if $|f| = |g|$ μ-a.e. □