Corollary 3.24

Let $(f_n)_{n\geq 1}$ be a sequence of measurable functions bounded above by an integrable function. Then

$$\limsup_{n\to\infty} \int f_n \, d\mu \leq \int \limsup_{n\to\infty} f_n \, d\mu .$$

Proof.

Apply Fatou's Lemma to $(-f_n)_{n\geq 1}$. $\square$

Proof of the Dominated Convergence Theorem

We are given $(f_n)_{n\geq 1}$ — a sequence of measurable functions converging to f μ-a.e. We have $|f_n(x)| \leq g(x)$ μ-a.e. x, where $g \geq 0$ is integrable.

We are to show that f_n, f are integrable and

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

Now the fact that f_n, f are integrable follows from 3.21 (ii) since $|f_n| \leq g$ μ a.e. and (hence) $|f| \leq g$ μ-a.e. (The fact that f is measurable follows from 3.13 (iii) and 3.12 (ii).)

We first get rid of the "μ-a.e." by defining modified functions

$h_n := f_n \cdot \chi_E$ and $h = f \cdot \chi_E$ where

$$E := \{ x : \lim_{n\to\infty} f_n(x) = f(x) \text{ and } |f_n(x)| \leq g(x) \text{ for all } n \geq 1 \}$$

Then $h_n = f_n$ μ-a.e. and $h = f$ μ-a.e., and now $\lim_{n\to\infty} h_n(x) = h(x)$ for all $x \in [0,1]$. Also, by 3.21, h, h_n are integrable and $\int f \, d\mu = \int h \, d\mu$, $\int f_n \, d\mu = \int h_n \, d\mu$ (for all $n \geq 1$), so it suffices to show that

$$\lim_{n\to\infty} \int h_n \, d\mu = \int h \, d\mu .$$

We show $\displaystyle\lim_{n\to\infty} \left| \int h_n \, d\mu - \int h \, d\mu \right| = 0$ (which suffices).

Now

$$\left| \int h_n \, d\mu - \int h \, d\mu \right| = \left| \int (h_n - h) \, d\mu \right| \qquad (3.20)$$

$$\leq \int |h_n - h| \, d\mu \qquad (3.19 \text{ (i)}).$$

so it suffices to show that $\int |h_n - h| \, d\mu \to 0$ as $n \to \infty$.

Now for $x \in [0,1]$, $\lim_{n \to \infty} |h_n(x) - h(x)| = 0$ and

$$0 \le |h_n(x) - h(x)| \le |h_n(x)| + |h(x)| = |f_n(x) \cdot \chi_E(x)| + |f(x) \cdot \chi_E(x)|$$
$$\le |f_n(x)| + |f(x)|$$
$$\le 2g(x).$$

So $|h_n - h|$ is bounded above and below by integrable functions.

So by Fatou's Lemma and its corollary, we have

$$\limsup_{n \to \infty} \int |h_n - h| \, d\mu \le \int \limsup_{n \to \infty} |h_n - h| \, d\mu$$
$$= \int 0 \, d\mu = 0 \qquad \text{(by 3.13')}$$
$$= \int \liminf_{n \to \infty} |h_n - h| \, d\mu \qquad \text{(by 3.13')}$$
$$\le \liminf_{n \to \infty} \int |h_n - h| \, d\mu$$
$$\le \limsup_{n \to \infty} \int |h_n - h| \, d\mu.$$

Thus all the terms are equal and we have

$$\lim_{n \to \infty} \int |h_n - h| \, d\mu = 0 \qquad \text{(by 3.13')}.$$

as required. $\square$

<u>Note</u>

If $E \subseteq \mathbb{R}$ is any measurable set, then everything above works for functions $f : E \to \mathbb{R}^*$ in place of functions $f : [0,1] \to \mathbb{R}^*$.

<u>Lebesgue integration versus Riemann integration</u>

The Lebesgue integral is really only defined to give as general a theory of integration as is possible and, in particular, to make the limit theorems, such as the Dominated Convergence Theorem, work. When it comes to actually calculating the integral of a particular function, however, we usually have to resort to the techniques of Riemann integration (and then, possibly, take limits as permitted by the Dominated Convergence Theorem). So we should show that if a function $f:[0,1]\to\mathbb{R}$ is Riemann integrable then it is integrable in our new sense - which we call <u>Lebesgue integrable</u>.

<u>Theorem 3.28</u>*

Let $f:[0,1]\to\mathbb{R}$ be Riemann integrable. Then f is Lebesgue integrable and $\int f\,d\mu = \int_0^1 f(x)\,dx$.

First a lemma which is a standard fact about Riemann integration (though possibly not proved in year 2).

<u>Theorem 3.27</u>

Let $f:[0,1]\to\mathbb{R}$ be (bounded and) Riemann integrable. Then $\int_0^1 f(x)\,dx$ may be calculated by use of the partitions $D_k := \{ i/2^k : i=0,\dots,2^k\}$. I.e. if we set

$$m_i^{(k)} := \inf\left\{f(x) : \tfrac{i}{2^k} \le x \le \tfrac{i+1}{2^k}\right\} \text{ and } M_i^{(k)} := \sup\left\{f(x) : \tfrac{i}{2^k} \le x \le \tfrac{i+1}{2^k}\right\}$$

and $L(f;D_k) := \sum_{i=0}^{2^k-1} m_i^{(k)}/2^k$ and $U(f,D_k) := \sum_{i=0}^{2^k-1} M_i^{(k)}/2^k$, then

$$\lim_{k\to\infty} L(f,D_k) = \lim_{k\to\infty} U(f,D_k) = \int_0^1 f(x)\,dx. \qquad \square$$

<u>Proof of 3.28</u>

Let D_k be the partition as above. Define simple function f_k, g_k by

$$f_k = \sum_{i=0}^{2^k-1} m_i^{(k)} \chi_{[i/2^k,\frac{i+1}{2^k})} + f(1)\cdot\chi_{\{1\}}, \quad g_k = \sum_{i=0}^{2^k-1} M_i^{(k)} \chi_{[i/2^k,\frac{i+1}{2^k})} + f(1)\chi_{\{1\}}$$

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* The numbering is as in the printed notes.