So it follows from the previous section that $\int f^+ d\mu$, $\int f^- d\mu$ and $\int |f|$ are all defined. It is tempting to define

$\int f\, d\mu := \int f^+ d\mu - \int f^- d\mu$, but the problem is if both the integrals on the right are infinite.

Now it follows from 3.14 that

$$\int |f|\, d\mu = \int f^+ d\mu + \int f^- d\mu.$$

Hence $\int |f|\, d\mu < \infty \iff \int f^+ d\mu < \infty$ and $\int f^- d\mu < \infty$.

<u>Definition</u>

If $\int |f|\, d\mu < \infty$ then we say that $f$ is <u>integrable</u> (<u>Lebesgue integrable</u>) and we define

$$\int f\, d\mu = \int f^+ d\mu - \int f^- d\mu.$$

(If $f$ is not integrable then we define $\int f\, d\mu$ to be $+\infty$ if $\int f^- d\mu$ is finite and to be $-\infty$ if $\int f^+ d\mu$ is finite. But $\int f\, d\mu$ is left undefined if both $\int f^+ d\mu$ and $\int f^- d\mu$ are infinite.)

<u>Definition</u>

$\mathcal{L}^1([0,1], \mu) :=$ the set of all integrable functions $f : [0,1] \to \mathbb{R}^*$.

<u>Lemma 3.19</u>

Let $f : [0,1] \to \mathbb{R}^*$ be measurable. Then $f$ is integrable if and only if $|f|$ is integrable, and if $f$ is integrable then

(i) $\left| \int f\, d\mu \right| \leq \int |f|\, d\mu$;

(ii) $f$ is finite valued $\mu$-a.e. on $[0,1]$, i.e. $\{x \in [0,1] : f(x) = \infty \text{ or } f(x) = -\infty\}$ is a null set.

<u>Proof.</u>

The first statement follows immediately from the definition of integrability.

Proof of (i): exercise.

(ii) Say $E_1 := \{x \in [0,1] : f(x) = \infty\}$ is not null.

Then $\mu(E_1) > 0$.

Now for all $n \geq 1$, $f^+ \geq n\chi_{E_1}$ (since if $\chi_{E_1}(x) = 1$, then $f^+(x) = \infty \geq n \cdot \chi_{E_1}(x)$, and if $\chi_{E_1}(x) = 0$ then $f^+(x) \geq 0 = n \cdot 0$).

Thus by Lemma 3.17 (iii),

$$\int f^+ d\mu \geq \int n\chi_{E_1} d\mu = n \int \chi_{E_1} d\mu = n \cdot \mu(E_1)$$ (by defn. of the integral of simple functions).

But this is true for all $n$, so $\int f^+ d\mu = \infty$ which contradicts (using (i)) the fact that $f^+$ is integrable. So $E_1$ is null.

Similarly, $E_2 := \{x \in [0,1] : f(x) = -\infty\}$ is null.

So $E_1 \cup E_2$ is null, as required. $\square$

<u>Proposition 3.20</u>

If $f, g$ are integrable and $c \in \mathbb{R}$, then $cf$ and $f+g$ (if defined) are integrable, and then

$$\int f+g \, d\mu = \int f \, d\mu + \int g \, d\mu \text{ and}$$

$$\int cf \, d\mu = c \int f \, d\mu.$$

<u>Proof.</u>

Exercise $\square$

<u>Proposition 3.21</u>

Suppose that $f, g : [0,1] \to \mathbb{R}^*$ are measurable.

(i) If $f = g$ $\mu$-a.e. and $f$ is integrable, then $g$ is integrable and $\int f \, d\mu = \int g \, d\mu$.

(ii) If $f$ is integrable and $|g| \leq |f|$ $\mu$-a.e., then $g$ is integrable

(iii) If $f$ is integrable and $f \geq 0$ $\mu$-a.e., then $\int f \, d\mu = 0$ implies that $f = 0$ $\mu$-a.e. (So if $f > 0$ $\mu$-a.e, then $\int f \, d\mu > 0$.)

(iv) If $f, g$ are integrable and $f \geq g$ $\mu$-a.e., then $\int f d\mu \geq \int g \, d\mu$.

Proof (Non-examinable)

(i) If $f = g$ $\mu$-a.e then $f^+ = g^+$ $\mu$-a.e. and $f^- = g^-$ $\mu$-a.e.
One must now inspect the proof of the representation of non-negative measurable functions as limits of simple functions to see that this result holds for non-negative measurable functions. So $\int f^+ d\mu = \int g^+ d\mu$ and $\int f^- d\mu = \int g^- d\mu$ and the result follows for $f, g$.
(If you've looked at the proof of 3.15, observe that the sets $Q_{n,k}$ have the same measure for $f$ as they do for $g$, and hence $\int f_n \, d\mu = \int g_n \, d\mu$ for all $n$. So the approximating integrals of simple functions have the same limit.)

(ii) Let $E = \{x \in [0,1] : |g(x)| > |f(x)|\}$. Then $\mu(E) = 0$ (so also $E \in \mathcal{M}([0,1])$). Define $h : [0,1] \to \mathbb{R}^*$ by

$$h(x) = \begin{cases} g(x) & \text{if } x \notin E \\ f(x) & \text{if } x \in E \end{cases}.$$

Then $g = h$ $\mu$-a.e. so $h$ is measurable (by 3.12(iii)) and by (i) above it is sufficient to show $h$ is integrable. However, $|h(x)| \leq |f(x)|$ for <u>all</u> $x \in [0,1]$, so

$$\int |h| \, d\mu \leq \int |f| \, d\mu \quad \text{(by 3.17 (iii))}$$
$$< \infty \quad \text{(since } f \text{ is integrable).}$$

$\therefore$ $h$ is integrable as required.

(iii) By (i) we can change $f(x)$ to $0$ if $f(x)$ was negative and this doesn't change the problem. So we may assume that $f \geq 0$ everywhere.

So assume that $\int f \, d\mu = 0$.

Now $\{x \in [0,1] : f(x) > 0\} = \bigcup_{n=1}^{\infty} A_n$,

where $A_n := \{x \in [0,1] : f(x) > 1/n\}$.

Assume, for a contradiction, that for some $n$, $\mu(A_n) > 0$.

Then $f \geq \frac{1}{n} \chi_{A_n}$ (everywhere), so that

$$\int f \, d\mu \geq \int \frac{1}{n} \cdot \chi_{A_n} \, d\mu \qquad \text{(by 3.17 (iii))}$$

$$= \frac{1}{n} \cdot \mu(A_n) > 0, \quad \text{contradiction.}$$

Hence $\mu(A_n) = 0$ for all $n$, so $\mu(\{x \in [0,1] : f(x) > 0\}) = 0$

(by Example sheet 3, exc. 3 (b) – or, rather, its solution).

So (as $f \geq 0$), $f = 0$ $\mu$-a.e.

(iv) By (i) we may adjust $g$ on a set of measure $0$ so that $f \geq g$ everywhere.

Then $\int (f-g) \, d\mu = \int f \, d\mu - \int g \, d\mu$ (by 3.20),

and $\int (f-g) \, d\mu \geq 0$ (by 3.17 (iii)), so $\int f \, d\mu \geq \int g \, d\mu$

as required. $\square$

## Limit Theorems

First, an example:

define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \begin{cases} n^2 & \text{if } 0 < x \leq 1/n \\ 0 & \text{otherwise} \end{cases}$

Clearly for every $x \in \mathbb{R}$, $\lim_{n \to \infty} f_n(x) = 0$. However,

$$\int f_n \, d\mu = \int n^2 \chi_{[0, 1/n]} \, d\mu = n^2 \cdot \mu((0, 1/n]) = n^2 \cdot \frac{1}{n} = n.$$

Hence, it is not true in general that for integrable $f_n, f$ that if $\lim_{n \to \infty} f_n = f$ then $\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu$. However:

<u>Theorem 3.22</u> (Monotone Convergence Theorem)

Suppose that $(f_n)_{n\geq 1}$ is an increasing sequence of non-negative, measurable functions. Write $f = \lim_{n\to\infty} f_n$ (which is well-defined, but may take infinite values). Then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

<u>Proof.</u>

For each $n \geq 1$, choose an increasing sequence $f_{n,k}$ of non-negative simple functions converging to $f_n$ as $k \to \infty$.

Set $g_k := \max_{n \leq k} f_{n,k}$. Then $(g_k)_{k\geq 1}$ is an increasing sequence of non-negative simple functions. Set $g := \lim_{k\to\infty} g_k$.

For $1 \leq n \leq k$,

$$f_{n,k} \leq g_k \leq f_k \leq f, \quad \ldots\ldots (*)$$

so letting $k \to \infty$ gives, for all $n \geq 1$,

$$f_n \leq g \leq f$$

and letting $n \to \infty$, gives $f = g$.

Now all the functions in $(*)$ are non-negative and measurable, so by 3.17 (iii) we have, for $1 \leq n \leq k$, that

$$\int f_{n,k} \, d\mu \leq \int g_k \, d\mu \leq \int f_k \, d\mu. \quad (**)$$

But by definition $\lim_{k\to\infty} \int f_{n,k} \, d\mu = \int f_n \, d\mu$ and $\lim_{k\to\infty} \int g_k \, d\mu = \int g \, d\mu$.

So $\int f_n \, d\mu \leq \int g \, d\mu \leq \lim_{k\to\infty} \int f_k \, d\mu$ for $n \geq 1$ (by taking $\lim_{k\to\infty}$ in $(**)$).

Now let $n \to \infty$ to obtain

$$\lim_{n\to\infty} \int f_n \, d\mu \leq \int g \, d\mu \leq \lim_{k\to\infty} \int f_k \, d\mu.$$

Since $f = g$, this is the required result. $\square$

The next result considers a sequence of (not necessarily non-negative) measurable functions and is the main Convergence Theorem for the sequence of integrals.

Theorem 3.26 (Dominated (or Lebesgue) Convergence Theorem)
Let $(f_n)_{n\geq 1}$ be a sequence of measurable functions and assume that $f_n(x)$ converges to $f(x)$ for $\mu$-a.e. $x$. Suppose there exists an integrable function $g \geq 0$ such that for all $n \geq 1$, $|f_n(x)| \leq g(x)$ for $\mu$-a.e. $x$.

Then $f_n$, $f$ are integrable and

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

For the proof, we first required the following

Theorem 3.23 (Fatou's Lemma)
Let $(f_n)_{n\geq 1}$ be a sequence of measurable functions which is bounded below by an integrable function $g$. (I.e. $\forall x \in [0,1]$, $\forall n \geq 1$, $f_n(x) \geq g(x)$.) Then $\int \liminf_{n\to\infty} f_n \, d\mu$ and $\liminf_{n\to\infty} \int f_n \, d\mu$ are both well-defined and

$$\int \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu.$$

Proof.
First note that if $h : [0,1] \to \mathbb{R}^*$ is any measurable function and $h \geq F$ for some integrable function $F : [0,1] \to \mathbb{R}^*$, then $\int h \, d\mu$ is well-defined (possibly $\infty$). This is because $0 \leq h^- \leq |F|$, so $\int h^- d\mu$ is finite (by 3.21(ii)). In particular, each $\int f_n \, d\mu$ is well-defined and since also $\liminf_{n\to\infty} f_n \geq g$ we have that $\int \liminf_{n\to\infty} f_n \, d\mu$ is well-defined.
Now, let $h_n = f_n - g$. Then $h_n$ is non-negative and measurable.
Set $g_n := \inf_{k\geq n} h_k$. Then $g_n$ is an increasing sequence of non-negative, measurable functions (see 3.13(i)) so, by the Monotone Convergence Theorem,

$$\int \lim_{n\to\infty} g_n \, d\mu = \lim_{n\to\infty} \int g_n \, d\mu \quad \text{---- } (*)$$

Now

$$\int \liminf_{n\to\infty} f_n \, d\mu = \int \left( \liminf_{n\to\infty} (h_n + g) \right) d\mu \qquad (\text{def. of } h_n)$$

$$= \int \left( \liminf_{n\to\infty} h_n + g \right) d\mu \qquad (\text{see below})^{(+)}$$

$$= \int \lim_{n\to\infty} g_n \, d\mu + \int g \, d\mu \qquad (\text{by } 3.20)$$

$$= \lim_{n\to\infty} \int g_n \, d\mu + \int g \, d\mu \qquad (\text{by } (*))$$

$$= \lim_{n\to\infty} \left( \int \inf_{k\geq n} h_k \, d\mu \right) + \int g \, d\mu \qquad (\text{def. of } g_n)$$

$$= \lim_{n\to\infty} \left( \int \inf_{k\geq n} (h_k + g) \, d\mu \right) \qquad (\text{by } 3.20).$$

Now, for $k \geq n$, $\inf_{k\geq n} (h_k + g) \leq (h_k + g)$ and so

$$\int \inf_{k\geq n} (h_k + g) \, d\mu \leq \int (h_k + g) \, d\mu \text{ , and hence}$$

$$\int \inf_{k\geq n} (h_k + g) \, d\mu \leq \inf_{k\geq n} \int (h_k + g) \, d\mu .$$

Combining this with the above, we get

$$\int \liminf_{n\to\infty} f_n \, d\mu \leq \lim_{n\to\infty} \left( \inf_{k\geq n} \int (h_k + g) \, d\mu \right)$$

$$= \liminf_{n\to\infty} \left( \int (h_n + g) \, d\mu \right.$$

$$= \liminf_{n\to\infty} \int f_n \, d\mu \quad (\text{def. of } h_n).$$

as required. $\square$

(+) Here we use the following exercise:

For any sequence $(a_n)_{n\geq 1}$ of real numbers and any $a \in \mathbb{R}$,

$$\liminf_{n\to\infty} (a_n + a) = \left( \liminf_{n\to\infty} a_n \right) + a .$$