So it follows from the previous section that $\int f^+ \, dm$, $\int f^- \, dm$ and $\int |f| \, dm$ are all defined. It is tempting to define

$$\int f \, dm := \int f^+ \, dm - \int f^- \, dm,$$

but the problem is if both the integrals on the right are infinite.

Now it follows from 3.14 that

$$\int |f| \, dm = \int f^+ \, dm + \int f^- \, dm.$$

Hence, $\int |f| \, dm < \infty \iff \int f^+ \, dm < \infty$ and $\int f^- \, dm < \infty$.

**Definition**

If $\int |f| \, dm < \infty$ then we say that $f$ is integrable (Lebesgue integrable) and we define

$$\int f \, dm := \int f^+ \, dm - \int f^- \, dm,$$

(If $f$ is not integrable then we define $\int f \, dm$ to be $+\infty$ if $\int f^- \, dm$ is finite and to be $-\infty$ if $\int f^+ \, dm$ is finite. But $\int f \, dm$ is left undefined if both $\int f^- \, dm$ and $\int f^+ \, dm$ are infinite.)

**Definition**

$L^1([0,1], \mu) :=$ the set of all integrable functions $f : [0,1] \to \mathbb{R}^+.$

**Lemma 3.19**

Let $f : [0,1] \to \mathbb{R}^+$ be measurable. Then $f$ is integrable if and only if $\int |f| \, dm$ is integrable, and if $f$ is integrable then

(i) $\int |f| \, dm \leq \int |f| \, dm$;

(ii) if $f$ is finite valued $\mu$-a.e. on $[0,1]$, i.e.

$\{x \in [0,1] : f(x) = +\infty \text{ or } f(x) = -\infty \}$ is a null set.

**Proof:**

The first statement follows immediately from the
definition of integrability.

Proof of (i): Exercise.

(ii) Say \( E_1 = \{ x \in [0,1] : f(x) = \infty \} \) is not null.

Then \( \mu(E_1) > 0 \).

Now for all \( n \geq 1 \), \( f^+ \geq n \chi_{E_1} \) (since if \( \chi_{E_1}(x) = 1 \), then 
\( f^+(x) = \infty \geq n \cdot \chi_{E_1}(x) \), and if \( \chi_{E_1}(x) = 0 \) then \( f^+(x) \geq 0 = n \cdot 0 \)).

Thus by Lemma 3.17(ii),

\[
\int f^+ \, d\mu \geq \int n \chi_{E_1} \, d\mu = n \int \chi_{E_1} \, d\mu = n \cdot \mu(E_1) \quad \text{(by defn. of integral of simple functions)}.
\]

But this is true for all \( n \), so \( \int f^+ \, d\mu = \infty \) which contradicts (using (i)) the fact that \( f^+ \) is integrable. So \( E_1 \) is null.

Similarly, \( E_2 = \{ x \in [0,1] : f(x) = -\infty \} \) is null.

So \( E_1 \cup E_2 \) is null, as required.

\[ \square \]

**Proposition 3.20**

If \( f, g \) are integrable and \( x \in \mathbb{R} \), then \( cf \) and \( f+g \)
(if defined) are integrable, and then

\[
\int (cf + g) \, d\mu = c \int f \, d\mu + \int g \, d\mu \quad \text{and} \quad \int c f \, d\mu = c \int f \, d\mu.
\]

**Proof**

Exercise

\[ \square \]

**Proposition 3.21**

Suppose that \( f, g : [0,1] \to \mathbb{R}^* \) are measurable.

(i) If \( f = g \) \( \mu \)-a.e. and \( f \) is integrable, then \( g \) is integrable
and \( \int f \, d\mu = \int g \, d\mu \).

(ii) If \( f \) is integrable and \( |g| \leq |f| \) \( \mu \)-a.e., then \( g \) is integrable.
(iii) If \( f \) is integrable and \( f \geq 0 \) \( \mu \)-a.e., then \( \int f \, d\mu = 0 \) implies that \( f = 0 \) \( \mu \)-a.e. (So if \( f \geq 0 \) \( \mu \)-a.e., then \( \int f \, d\mu = 0 \).)

(iv) If \( f, g \) are integrable and \( f \geq g \) \( \mu \)-a.e., then \( \int f \, d\mu \geq \int g \, d\mu \).

Proof (Non-examinable)

(i) If \( f = g \) \( \mu \)-a.e. then \( f^+ = g^+ \) \( \mu \)-a.e. and \( f^- = g^- \) \( \mu \)-a.e.

One must now inspect the proof of the representation of non-negative measurable functions as limits of simple functions to see that this result holds for non-negative measurable functions. So \( \int f^+ \, d\mu = \int g^+ \, d\mu \) and \( \int f^- \, d\mu = \int g^- \, d\mu \) and the result follows for \( f, g \).

(If you've looked at the proof of 3.15, observe that the sets \( A_{\epsilon,k} \) have the same measure for \( f \) as they do for \( g \), and hence \( \int f \, d\mu = \int g \, d\mu \) for all \( n \). So the approximating integrals of simple functions have the same limit.)

(iii) Let \( E = \{ x \in [0,1] : |g(x)| > |f(x)| \} \). Then \( \mu(E) = 0 \) (so also \( E \in \mathcal{M}([0,1]) \)). Define \( h : [0,1] \to \mathbb{R}^+ \) by

\[
h(x) = \begin{cases} 
g(x) & \text{if } x \notin E \\
g(x) & \text{if } x \in E
\end{cases}
\]

Then \( g = h \) \( \mu \)-a.e. so \( h \) is measurable (by 3.12(iii)) and by (i) above it is sufficient to show \( h \) is integrable.

However, \( |h(x)| \leq |f(x)| \) for all \( x \in [0,1] \), so

\[
\int |h| \, d\mu \leq \int |f| \, d\mu \quad \text{(by 3.17 (iii))}
\]

\[
< \infty \quad \text{(since \( f \) is integrable)}
\]

\( h \) is integrable as required.
By (i) we can change \( f(x) \) to 0 if \( f(x) \) was negative and this doesn't change the problem. So we may assume that \( f \geq 0 \) everywhere.

So assume that \( \int f \, d\mu = 0 \).

Now \( \{ x \in [0,1] : f(x) > 0 \} = \bigcup_{n=1}^{\infty} A_n \),

where \( A_n = \{ x \in [0,1] : f(x) > \frac{1}{n^2} \} \).

Assume, for a contradiction, that for some \( n \), \( \mu(A_n) > 0 \).

Then \( f \geq \frac{1}{n^2} \chi_{A_n} \) (everywhere), so that

\[
\int f \, d\mu \geq \int \frac{1}{n^2} \chi_{A_n} \, d\mu \quad \text{(by 3.17 (iii))}
\]

\[
= \frac{1}{n^2} \mu(A_n) > 0 \quad \text{contradiction.}
\]

Hence \( \mu(A_n) = 0 \) for all \( n \), so \( \mu(\{ x \in [0,1] : f(x) > 0 \}) = 0 \)

(by Example sheet 3, exc. 3 (6) - or, rather, its solution).

So - (as \( f \geq 0 \)), \( f = 0 \, \mu\text{-a.e.} \).

(iv) By (i) we may adjust \( g \) on a set of measure 0 so that \( f \geq g \) everywhere.

Then \( \int (f-g) \, d\mu = \int f \, d\mu - \int g \, d\mu \quad \text{(by 3.20)} \),

and \( \int (f-g) \, d\mu \geq 0 \quad \text{(by 3.17 (iii))}, \) so \( \int f \, d\mu \geq \int g \, d\mu \) as required.

\[ \square \]

**Limit Theorems**

First, an example:

Define \( f_n : \mathbb{R} \rightarrow \mathbb{R} \) by \( f_n(x) = \begin{cases} n^2 & \text{if } 0 < x \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases} \)

Clearly for every \( x \in \mathbb{R} \), \( \lim_{n \to \infty} f_n(x) = 0 \). However,

\[
\int f_n \, d\mu = \int n^2 \chi_{(0,\frac{1}{n})} \, d\mu = n^2 \cdot \mu((0,\frac{1}{n})) = n^2 \cdot \frac{1}{n} = n.
\]

Hence, it is not true in general that for integrable \( f_n, f \) that if \( \lim_n f_n = f \) then \( \lim_n \int f_n \, d\mu = \int f \, d\mu \). However:
Theorem 3.22 (Monotone Convergence Theorem)

Suppose that \((f_n)_{n \geq 1}\) is an increasing sequence of non-negative, measurable functions. Write \(f = \lim_{n \to \infty} f_n\) (which is well-defined, but may take infinite values). Then

\[
\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu.
\]

Proof.

For each \(n \geq 1\), choose an increasing sequence \(f_{n,k}\) of non-negative simple functions converging to \(f_n\) as \(k \to \infty\).

Set \(g_k := \max_{n \leq k} f_{n,k}\). Then \((g_k)_{k \geq 1}\) is an increasing sequence of non-negative simple functions. Set \(g := \lim_{k \to \infty} g_k\).

For \(1 \leq n \leq k\),

\[
f_{n,k} \leq g_k \leq f_k \leq S,
\]

so letting \(k \to \infty\) gives, for all \(n \geq 1\),

\[
f_n \leq g \leq S,
\]

and letting \(n \to \infty\) gives \(f = g\).

Now all the functions in \((x)\) are non-negative and measurable, so by 3.12 (iii) we have, for \(1 \leq n \leq k\), that

\[
\int f_{n,k} \, d\mu \leq \int g_k \, d\mu \leq \int f_k \, d\mu.
\]

But by definition \(\lim_{k \to \infty} \int f_{n,k} \, d\mu = \int f_n \, d\mu\) and \(\lim_{k \to \infty} \int g_k \, d\mu = \int g \, d\mu\).

So \(\int f_n \, d\mu \leq \int g \, d\mu \leq \lim_{k \to \infty} \int g_k \, d\mu\) for \(n \geq 1\) (by taking \(\lim_{k \to \infty}\) in \((xy)\)).

Now let \(n \to \infty\) to obtain

\[
\lim_{n \to \infty} \int f_n \, d\mu \leq \int g \, d\mu \leq \lim_{k \to \infty} \int g_k \, d\mu.
\]

Since \(f = g\), this is the required result. \(\square\)
The next result concerns a sequence of (not necessarily non-negative) measurable functions, and is the main Convergence Theorem for the sequence of integrals.

**Theorem 3.26 (Dominated (or Lebesgue) Convergence Theorem)**

Let \((f_n)_{n \geq 1}\) be a sequence of measurable functions and assume that \(f_n(x)\) converges to \(f(x)\) for \(\mu\)-a.e. \(x\). Suppose there exists an integrable function \(g \geq 0\) such that for all \(n \geq 1\),

\[ |f_n(x)| \leq g(x) \quad \text{for } \mu\)-a.e. \(x\).

Then \(f_n, f\) are integrable and

\[ \lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu. \]

For the proof, we first require the following

**Theorem 3.23 (Fatou's Lemma)**

Let \((f_n)_{n \geq 1}\) be a sequence of measurable functions, which is bounded below by an integrable function \(g\). (i.e., \(\forall x \in [0,1] : f_n(x) \geq g(x)\).) Then \(\liminf_{n \to \infty} f_n \) and \(\limsup_{n \to \infty} f_n \) are both well-defined and

\[ \int \liminf_{n \to \infty} f_n \, d\mu \leq \liminf_{n \to \infty} \int f_n \, d\mu. \]

**Proof.**

First note that if \(h : [0,1] \to \mathbb{R}^+\) is any measurable function and \(h \geq F\) for some integrable function \(F : [0,1] \to \mathbb{R}^+\), then \(\int h \, d\mu\) is well-defined (possibly \(\infty\)). This is because

\[ 0 \leq h \leq |F|, \quad \text{so } \int h \, d\mu \text{ is finite} \quad \text{(by 3.21(ii)).} \]

In particular, each \(\int f_n \, d\mu\) is well-defined and since also \(\liminf_{n \to \infty} f_n \geq g\), we have that \(\int \liminf_{n \to \infty} f_n \, d\mu\) is well-defined.

Now, let \(h_n = f_n - g\). Then \(h_n\) is non-negative and measurable.

Set \(g_n := \inf_{k \geq n} h_k\). Then \(g_n\) is an increasing sequence of non-negative measurable functions (see 3.13(i)) \(\to g\) , by the Monotone Convergence Theorem,
\[
\int \lim g_n \, d\mu = \lim_{n \to \infty} \int g_n \, d\mu \quad \cdots \quad (\ast)
\]

Now,

\[
\int \liminf_{n \to \infty} f_n \, d\mu = \int \left( \liminf_{n \to \infty} (h_n + g) \right) \, d\mu \quad \text{(def. of } h_n) \]

\[= \int \left( \lim_{n \to \infty} g_n + g \right) \, d\mu \quad \text{(see below)} (\dagger) \]

\[= \int \lim g_n \, d\mu + \int g \, d\mu \quad \text{(by 3.20)} \]

\[= \lim_{n \to \infty} \int g_n \, d\mu + \int g \, d\mu \quad \text{(by 3.20)} \]

\[= \lim_{n \to \infty} \left( \int \inf_{k \geq n} h_k \, d\mu \right) + \int g \, d\mu \quad \text{(def. of } g_n) \]

\[= \lim_{n \to \infty} \left( \int \inf_{k \geq n} (h_k + g) \, d\mu \right) \quad \text{(by 3.20)} \]

Now, for \( k \in \mathbb{N} \), \( \inf_{k < n} (h_k + g) \leq (h_n + g) \) and \( \infty \)

\[
\int \inf_{k \geq n} (h_k + g) \, d\mu \leq \int (h_n + g) \, d\mu \quad \text{and hence}
\]

\[
\int \inf_{k < n} (h_k + g) \, d\mu \leq \inf_{k < n} \int (h_k + g) \, d\mu.
\]

Combining this with the above, we get

\[
\int \liminf_{n \to \infty} f_n \, d\mu \leq \lim_{n \to \infty} \left( \inf_{k \geq n} \int (h_k + g) \, d\mu \right)
\]

\[= \lim_{n \to \infty} \left( \inf_{k \geq n} \int (h_k + g) \, d\mu \right) \]

\[= \liminf_{n \to \infty} \int f_n \, d\mu \quad \text{(def. of } h_n) \]

as required.

(\dagger) Here we use the following exercise:

For any sequence \((a_n)_{n \geq 1}\) of real numbers and any \(a \in \mathbb{R}\),

\[
\liminf_{n \to \infty} (a_n + a) = \left( \liminf_{n \to \infty} a_n \right) + a.
\]