Now suppose $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = l \in \mathbb{R}$, say.

Let $\varepsilon > 0$ be given. Then we may choose N so that

for all $n \geq N$, $|\sup_{k\geq n} x_k - l| < \varepsilon$ and $|\inf_{k\geq n} x_k - l| < \varepsilon$.

In particular, $\sup_{k\geq N} x_k < l+\varepsilon$ and $\inf_{k\geq N} x_k > l-\varepsilon$.

So for all $k \geq N$, $x_k < l+\varepsilon$ and $x_k > l-\varepsilon$.

So $\forall k \geq N$, $|x_k - l| < \varepsilon$, so $\lim_{k\to\infty} x_k = l$.

If $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \infty$, then given any M, there exists N such that $\forall n \geq N$, $\inf_{k\geq n} x_k > M$. In particular, $\forall k \geq N$, $x_k > M$. This shows $x_k \to \infty$ as $k \to \infty$, as required. The case of $-\infty$ is left as an exercise. $\square$

<u>Proof of Lemma 3.13 (ii)</u> 23/10

Suppose that $\forall x \in [0,1]$, $\lim_{n\to\infty} f_n(x) = f(x)$ and each f_n is measurable.

By 3.13′, $\limsup_{n\to\infty} f_n(x) = f(x)$ $(\forall x \in [0,1])$.

I.e. $\lim_{n\to\infty} \left(\sup_{k\geq n} f_k(x)\right) = f(x)$.

For each $n \geq 1$, let $g_n = \sup_{k\geq n} f_k$. Then each g_n is measurable by 3.13 (i). Also, for each $x \in [0,1]$, $(g_n(x))_{n\geq 1}$ is a decreasing sequence.

Hence $f = \lim_{n\to\infty} g_n = \inf_{n\geq 1} g_n$. By 3.13 (i) again, $\inf_{n\geq 1} g_n$ is measurable, i.e. f is measurable. $\square$

Proposition 3.14

Let $f, g : [0,1] \to \mathbb{R}^*$ be measurable and let $c \in \mathbb{R}$. Then cf, $f+g$, $f \cdot g$ (if defined) are measurable.

Proof.

First note that the constant function zero is measurable (by 3.12). So to show cf is measurable, assume $c \neq 0$. Then

$$\{x \in [0,1] : cf(x) \leq a\} = \begin{cases} \{x \in [0,1] : f(x) \leq a/c\} & \text{if } c > 0 \\ \{x \in [0,1] : f(x) \geq a/c\} & \text{if } c < 0. \end{cases}$$

So cf is measurable since f is.

Suppose f, g are both measurable.

Note that $f(x) + g(x) > a \iff \exists r \in \mathbb{Q}$ s.t. $f(x) > r$ and $g(x) > a - r$

Hence
$$(f+g)^{-1}((a,\infty]) = \{x \in [0,1] : f(x)+g(x) > a\}$$

$$= \bigcup_{r \in \mathbb{Q}} (\{x \in [0,1] : f(x) > r\} \cap \{x \in [0,1] : g(x) > a-r\})$$

$$= \bigcup_{r \in \mathbb{Q}} (f^{-1}((r,\infty]) \cap g^{-1}((a-r,\infty])) \in \mathcal{M}([0,1]).$$

(We won't keep repeating the fact that sets like this one lie in $\mathcal{M}([0,1])$ because $\mathcal{M}([0,1])$ is a σ-algebra.)

Finally, for $f \cdot g$ we first consider f^2. We have

$$\{x \in [0,1] : f^2(x) \leq a\} = \begin{cases} \emptyset & \text{if } a < 0 \\ \{x \in [0,1] : f(x) = 0\} & \text{if } a = 0 \\ \{x \in [0,1] : -\sqrt{a} \leq f(x) \leq \sqrt{a}\} & \text{if } a > 0. \end{cases}$$

All three sets are in $\mathcal{M}([0,1])$ (easy exercise), so f^2 is measurable. We now use the trick

$$f \cdot g = \tfrac{1}{2}((f+g)^2 - f^2 - g^2)$$

to deduce from the above that $f \cdot g$ is measurable. $\square$

Integrating non-negative measurable functions

(To keep the flow of the development of the Lebesgue integral running smoothly, I shall omit all proofs that are non-examinable. They are relegated to an appendix in the printed notes: I shall return to them if time permits.)

Theorem 3.15

Let $f: [0,1] \to \mathbb{R}^*$ be a non-negative (i.e. $\forall x \in [0,1]$, $f(x) \geq 0$) measurable function. Then there exists an increasing sequence of non-negative simple function f_n $(n \geq 1)$, such that f_n converges pointwise to f as $n \to \infty$.

Proof. Non-examinable. $\square$

Proposition 3.16

Let f_n, g_n, $n \geq 1$ be two increasing sequences of non-negative simple functions which converge pointwise to the same measurable function on $[0,1]$, as $n \to \infty$. Then

$$\lim_{n \to \infty} \int f_n \, d\mu = \lim_{n \to \infty} \int g_n \, d\mu.$$

Proof. Non-examinable. $\square$

Definition

Let $f: [0,1] \to \mathbb{R}^*$ be a non-negative measurable function. Then we define

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu$$

where f_n is some increasing sequence of non-negative simple functions converging pointwise to f (as given by 3.15).

(Notice that the $\int f_n \, d\mu$'s have already been defined (just before 3.8).)

By 3.16, $\int f \, d\mu$ is well-defined.

<u>Lemma 3.17</u>

If $f, g : [0,1] \to \mathbb{R}^*$ are non-negative measurable functions and $c \in \mathbb{R}^+$, then

(i) $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$;

(ii) $\int c f \, d\mu = c \int f \, d\mu$;

(iii) if $f \geq g$, then $\int f \, d\mu \geq \int g \, d\mu$.

<u>Proof.</u>

Exercises. (Hint for (iii): Consider $f - g$.) $\square$

Can now do all sheet 3 and sheet 4.

<u>Integrating general measurable functions.</u>

We must now consider measurable functions that can take both positive and negative values.

Let $f : [0,1] \to \mathbb{R}^*$ be any function.

Define $f^+ : [0,1] \to \mathbb{R}^*$ and $f^- : [0,1] \to \mathbb{R}^*$ by

$$f^+(x) = \max\{0, f(x)\} \text{ and } f^{-}(x) = \max\{0, -f(x)\}.$$

Then f^+ and f^- are both non-negative functions and we have

$$f = f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^- .$$

It also follows from 3.13(i) and 3.14 that:

<u>Lemma 3.18</u>

If $f : [0,1] \to \mathbb{R}^*$ is measurable, then f^+, f^- and $|f|$ are all measurable. $\square$