Now suppose \( \lim \sup_{n \to \infty} x_n = \lim \inf_{n \to \infty} x_n = l \in \mathbb{R} \), say.

Let \( \varepsilon > 0 \) be given. Then we may choose \( N \) so that for all \( n \geq N \),

\[
| \sup_{k \geq n} x_k - l | < \varepsilon \quad \text{and} \quad | \inf_{k \geq n} x_k - l | < \varepsilon .
\]

In particular,

\[
\sup_{k \geq N} x_k < l + \varepsilon \quad \text{and} \quad \inf_{k \geq N} x_k > l - \varepsilon .
\]

So for all \( k \geq N \), \( x_k < l + \varepsilon \) and \( x_k > l - \varepsilon \).

So \( \forall k \geq N \), \( | x_k - l | < \varepsilon \), so \( \lim_{k \to \infty} x_k = l \).

If \( \lim \sup_{n \to \infty} x_n = \lim \inf_{n \to \infty} x_n = \infty \), then given any \( M \), there exists \( N \) such that \( \forall n \geq N \), \( \inf_{k \geq n} x_k > M \). In particular, \( \forall k \geq N \), \( x_k > M \). This shows \( x_k \to \infty \) as \( k \to \infty \), as required. The case of \( -\infty \) is left as an exercise. \( \square \)

---

Proof of Lemma 3.13 (ii)

Suppose that \( \forall x \in [0,1] \), \( \lim_{n \to \infty} f_n(x) = f(x) \) and each \( f_n \) is measurable.

By 3.13', \( \lim \sup_{n \to \infty} f_n(x) = f(x) \) (\( \forall x \in [0,1] \)).

I.e. \( \lim_{n \to \infty} ( \sup_{k \geq n} f_k(x) ) = f(x) \).

For each \( n \geq 1 \), let \( g_n = \sup_{k \geq n} f_k \). Then each \( g_n \) is measurable by 3.13 (i). Also, for each \( x \in [0,1] \),

\[ (g_n(x))_{n \geq 1} \]

is a decreasing sequence.

Hence \( f = \lim_{n \to \infty} g_n = \inf_{n \geq 1} g_n \). By 3.13 (i) again,

\( \inf_{n \geq 1} g_n \) is measurable, i.e. \( f \) is measurable. \( \square \)
Proposition 3.14

Let \( f, g : [0,1] \to \mathbb{R}^+ \) be measurable and let \( c \in \mathbb{R} \). Then \( cf, f+g, fg \) (if defined) are measurable.

Proof.

First note that the constant function zero is measurable (by 3.12). So to show \( cf \) is measurable, assume \( c \neq 0 \).

Then

\[
\{ x \in [0,1] : cf(x) \leq a \} = \bigcup_{\epsilon > 0} \{ x \in [0,1] : f(x) \leq \frac{a}{c} + \frac{\epsilon}{|c|} \}.
\]

So \( cf \) is measurable since \( f \) is.

Suppose \( f, g \) are both measurable.

Note that \( f(x) + g(x) > a \) \( \iff \exists r \in \mathbb{R} \) s.t. \( f(x) > r \) and \( g(x) > a - r \).

Hence \((f+g)^{-1}(\mathbb{R}, \infty) = \{ x \in [0,1] : f(x) + g(x) > a \} \)

\[
= \bigcup_{r \in \mathbb{R}} (f^{-1}(r, \infty) \cap g^{-1}(a-r, \infty)) \in M([0,1]).
\]

(We won't keep repeating the fact that sets like this one lie in \( M([0,1]) \) because \( M([0,1]) \) is a \( \sigma \)-algebra.)

Finally, for \( f \cdot g \) we first reconsider \( f^2 \). We have

\[
\{ x \in [0,1] : f^2(x) \leq a \} = \begin{cases} \emptyset & \text{if } a < 0 \\ \{ x \in [0,1] : f(x) = \sqrt{a} \} & \text{if } a = 0 \\ \{ x \in [0,1] : -\sqrt{a} \leq f(x) \leq \sqrt{a} \} & \text{if } a > 0. 
\end{cases}
\]

All three sets are in \( M([0,1]) \) (easy exercise), so \( f^2 \) is measurable. We now use the trick

\[
f \cdot g = \frac{1}{2} ((f+g)^2 - f^2 - g^2)
\]

to deduce from the above that \( f \cdot g \) is measurable. \( \square \)
Integrating non-negative measurable functions

(To keep the flow of the development of the Lebesgue integral running smoothly, I shall omit all proofs that are non-examinable. They are relegated to an appendix in the printed notes; I shall return to them if time permits.)

Theorem 3.15

Let \( f : [0,1] \to \mathbb{R}^+ \) be a non-negative (i.e., \( \forall x \in [0,1], f(x) \geq 0 \)) measurable function. Then there exists an increasing sequence of non-negative simple functions \( f_n \) (\( n \geq 1 \)), such that \( f_n \) converges pointwise to \( f \) as \( n \to \infty \).

Proof. Non-examinable.

Proposition 3.16

Let \( f_n, g_n, n \geq 1 \) be two increasing sequences of non-negative simple functions which converge pointwise to the same measurable function on \([0,1]\), as \( n \to \infty \). Then

\[
\lim_{n \to \infty} \int f_n \, d\mu = \lim_{n \to \infty} \int g_n \, d\mu.
\]

Proof. Non-examinable.

Definition

Let \( f : [0,1] \to \mathbb{R}^+ \) be a non-negative measurable function. Then we define

\[
\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu
\]

where \( f_n \) is some increasing sequence of non-negative simple functions, converging pointwise to \( f \) (as given by 3.15).
(Notice that the $\int f \, d\mu$'s have already been redefined.
(just before 3.8).)

By 3.16, $\int f \, d\mu$ is well-defined.

**Lemma 3.17**

Let $f, g : [0, 1] \to \mathbb{R}^+$ be non-negative measurable functions and $c \in \mathbb{R}^+$, then

1. $\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$;
2. $\int cf \, d\mu = c \int f \, d\mu$;
3. if $f \geq g$, then $\int f \, d\mu \geq \int g \, d\mu$.

**Proof.**

Examine. (Hint for (iii): Consider $f - g$.)

---

**Integrating general measurable functions.**

We must now consider measurable functions that can take both positive and negative values.

Let $f : [0, 1] \to \mathbb{R}^*$ be any function.

Define $f^+ : [0, 1] \to \mathbb{R}^+$ and $f^- : [0, 1] \to \mathbb{R}^+$ by

$$f^+(x) = \max\{0, f(x)\} \quad \text{and} \quad f^-(x) = \max\{0, -f(x)\}.$$  

Then $f^+$ and $f^-$ are both non-negative functions, and so we have

$$f = f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^-.$$  

It also follows from 3.13(i) and 3.14 that:

**Lemma 3.18**

If $f : [0, 1] \to \mathbb{R}^*$ is measurable, then $f^+, f^-$ and $|f|$ are all measurable.