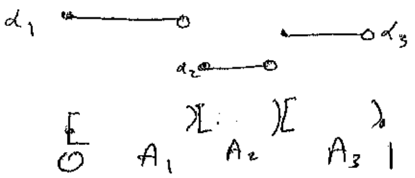
such that

$$f = \sum_{i=1}^{n} \alpha_i \cdot \chi_{A_i} .$$



Definition

Let $f : [0,1] \to \mathbb{R}$ be a simple function given as above. Then we define

$$\int f \, d\mu \;:=\; \sum_{i=1}^{n} \alpha_i \, \mu(A_i) .$$

Lemma 3.8

If f and g are simple functions and $k \in \mathbb{R}$, then kf, $|f|$, $f+g$ and $f \cdot g$ are simple functions.

Proof.

Exercise.

Lemma 3.9

If $f, g : [0,1] \to \mathbb{R}$ are simple functions and $a, b \in \mathbb{R}$ then

(i) $\int (af + bg) \, d\mu = a \int f \, d\mu + b \int g \, d\mu$;

(ii) if $f(x) \le g(x)$ for all $x \in [0,1]$ then

$$\int f \, d\mu \le \int g \, d\mu \; ;$$

(iii) $\left| \int f \, d\mu \right| \le \int |f| \, d\mu .$

Proof.

Exercises. $\square$

Measurable functions

Write $\mathbb{R}^* = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ (and use ∞ for $+\infty$).
For $y \in \mathbb{R}$, $y + \infty := \infty$, $y - \infty := -\infty$, $y \cdot \infty := \infty$ (for $y > 0$), but $\infty - \infty$ and $0 \cdot \infty$ are left undefined. We also set $[a, \infty] := [a, \infty) \cup \{+\infty\}$ etc, and $\sup(E)$, $\inf(E)$ for $E \subseteq \mathbb{R}^*$ are defined in obvious way.

We now introduce the class of functions that we will be able to integrate. They are those that satisfy one of the four equivalent conditions in the following lemma.

Lemma 3.10

Let $f : [0,1] \to \mathbb{R}^*$ be any function. Then the following statements are equivalent:

(i) $\forall a \in \mathbb{R}$, $f^{-1}([-\infty, a]) \in \mathcal{M}([0,1])$;

(ii) $\forall a \in \mathbb{R}$, $f^{-1}([-\infty, a)) \in \mathcal{M}([0,1])$;

(iii) $\forall a \in \mathbb{R}$, $f^{-1}([a, \infty]) \in \mathcal{M}([0,1])$;

(iv) $\forall a \in \mathbb{R}$, $f^{-1}((a, \infty]) \in \mathcal{M}([0,1])$.

[If $f : X \to Y$ is any function, and $A \subseteq Y$, then $f^{-1}(A)$ is defined to be the set $\{x \in X : f(x) \in A\}$. So, for example, in (i), $f^{-1}([-\infty, a]) = \{x \in [0,1] : f(x) \le a\}$, and in (ii), $f^{-1}([-\infty, a)) = \{x \in [0,1] : f(x) < a\}, \ldots$]

For the proof. you should first check the follow two facts that hold for any function $f : X \to Y$ between two sets:

(i) for any sequence of subsets $E_1, E_2, \ldots$ of Y,

$$f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(E_n);$$

(ii) for any set $E \subseteq Y$, $f^{-1}(Y \setminus E) = X \setminus (f^{-1}(E))$.

Proof of 3.10

(i) $\Rightarrow$ (ii): Assume (i). Then

$$f^{-1}([-\infty, a)) = f^{-1}\left(\bigcup_{n=1}^{\infty} [-\infty, a - \tfrac{1}{n}]\right) = \bigcup_{n=1}^{\infty} f^{-1}\left([-\infty, a - \tfrac{1}{n}]\right).$$

By (i), $f^{-1}([-\infty, a - \frac{1}{n}]) \in \mathcal{M}([0,1])$ for each n. But $\mathcal{M}([0,1])$ is a σ-algebra, so $\bigcup_{n=1}^{\infty} f^{-1}([-\infty, a - \frac{1}{n}]) \in \mathcal{M}([0,1])$, i.e. $f^{-1}([-\infty, a)) \in \mathcal{M}([0,1])$, so (ii) holds.

(ii) ⇒ (iii): Assume (ii).

Now $[a,\infty] = \mathbb{R}^* \setminus [-\infty, a)$. Thus

$$f^{-1}([a,\infty]) = f^{-1}(\mathbb{R}^* \setminus [-\infty,a)) = [0,1] \setminus f^{-1}([-\infty,a)),$$

which is in $\mathcal{M}([0,1])$ by (ii) and the fact that σ-algebras are closed under complementation. Thus (iii) holds.

(iii) ⇒ (iv).

This is similar to '(i) ⇒ (ii)'. Thus, assuming (iii) holds we have

$$f^{-1}((a,\infty]) = f^{-1}\Big(\bigcup_{n=1}^{\infty} [a+\tfrac{1}{n}, \infty]\Big) = \bigcup_{n=1}^{\infty} f^{-1}\big([a+\tfrac{1}{n}, \infty]\big)$$

which lies in $\mathcal{M}([0,1])$ by (iii) and the fact that $\mathcal{M}([0,1])$, being a σ-algebra, is closed under taking countable unions.

(iv) ⇒ (i).

This is similar to '(ii) ⇒ (iii)'. Thus, assuming (iv),

$$f^{-1}([-\infty,a]) = f^{-1}(\mathbb{R}^* \setminus (a,\infty]) = [0,1] \setminus f^{-1}((a,\infty]),$$

which is $\mathcal{M}([0,1])$ by (iv) and the fact that $\mathcal{M}([0,1])$ is closed under complementation. □

Definition

A function $f : [0,1] \to \mathbb{R}^*$ is called a <u>measurable function</u> if it satisfies one (and hence all) of the conditions in Lemma 3.10.

Lemma 3.11

(i) Simple functions are measurable.

(ii) Continuous functions are measurable.

Proof.

(i) Exercise.

(ii) Let $f : [0,1] \to \mathbb{R}^*$ be continuous. Assume first that $f : [0,1] \to \mathbb{R}$. We verify 3.10 (iv). Now $f^{-1}((a,\infty]) = f^{-1}((a,\infty))$, and the inverse image of an open set by a continuous function is open. So $f^{-1}((a,\infty)) = U \cap [0,1]$ for some open $U \subseteq \mathbb{R}$. But U may written as a countable union of open intervals ... $U \in \mathcal{B} \subseteq \mathcal{M}$ (Lemma 3.5). Also $[0,1] \in \mathcal{B}$ (since $[0,1] = \mathbb{R} \setminus ((-\infty,0) \cup (1,\infty))$, $\therefore U \cap [0,1] \in \mathcal{B} \subseteq \mathcal{M}$.

So $U \cap [0,1] \in \mathcal{M}([0,1])$.

If f takes the value ∞ (or $-\infty$) at some point, then it must be constantly ∞ (or $-\infty$), so $f^{-1}((a,\infty]) = [0,1]$ or $\emptyset$, both of which lie in $\mathcal{M}([0,1])$. $\square$

<u>Convention</u>

Let $P(x)$ be some property of the real number x (e.g. "x is not a rational number"). We say that P holds <u>almost everywhere</u> or, μ-almost everywhere (μ-a.e.), if the set $\{x \in \mathbb{R} : P(x) \text{ is false}\}$ is null.

Thus, since $\mathbb{Q}$ is countable, and hence null, it follows that $\chi_{\mathbb{Q}} = 0$ almost everywhere, where $\chi_{\mathbb{Q}}$ is the characteristic function of the rationals.

<u>Lemma 3.12</u>

(i) Suppose that $f : [0,1] \to \mathbb{R}^*$ is a function such that $f = 0$ almost everywhere. Then f is measurable.

(ii) Suppose that $f, g : [0,1] \to \mathbb{R}^*$ are two functions such that $f = g$ almost everywhere. Then f is measurable if and only if g is measurable.

<u>Proof.</u>

(i) We use (i) of 3.10, so let $a \in \mathbb{R}$ and consider $f^{-1}([-\infty,a])$. If $a < 0$, then $f^{-1}([-\infty,a])$ is a null set (as $f = 0$ μ-a.e and $f(x) \in [-\infty,a]$ implies $f(x) \neq 0$), and hence $f^{-1}([-\infty,a]) \in \mathcal{M}([0,1])$ (by definition of $\mathcal{M}([0,1])$). If $a \geq 0$, then similarly $(f^{-1}([-\infty,a]))^c$ is a null set, so $(f^{-1}([-\infty,a]))^c \in \mathcal{M}([0,1])$. But $\mathcal{M}([0,1])$ (being a σ-algebra) is closed under complementation so $\left(f^{-1}([-\infty,a])\right) \in \mathcal{M}([0,1])$.

(ii) Let $A = \{x \in [0,1] : f(x) \neq g(x)\}$, a null set.

Assume f is measurable. Let $a \in \mathbb{R}$. Then

$$g^{-1}([-\infty,a]) = (g^{-1}([-\infty,a]) \cap A) \cup (g^{-1}([-\infty,a]) \cap A^c) \quad \text{---- } (*)$$

A subset of a null set is null (obvious from the definition) so

$g^{-1}([-\infty,a]) \cap A$ is null, hence measurable.

But $g^{-1}([-\infty,a]) \cap A^c = f^{-1}([-\infty,a]) \cap A^c$ (defn. of A) ... $(**)$,

and as f is measurable, $f^{-1}([-\infty,a]) \in \mathcal{M}([0,1])$. Also, $\mathcal{M}([0,1])$ is closed under complementation, so $A^c \in \mathcal{M}([0,1])$.

As $\mathcal{M}([0,1])$ is also closed under intersection it follows from $(**)$ that $g^{-1}([-\infty,a]) \cap A^c \in \mathcal{M}([0,1])$.

Finally, as $\mathcal{M}([0,1])$ is closed under unions, it follows from $(*)$ that $g^{-1}([-\infty,a]) \in \mathcal{M}([0,1])$, as required. $\square$

Lemma 3.13

Let $f_n : [0,1] \to \mathbb{R}^*$, $n \geq 1$, be a sequence of measurable functions. Then

(i) $\sup_n f_n$ and $\inf_n f_n$ are measurable;

(ii) if f_n converges to f pointwise as $n \to \infty$ (i.e. if for each $x \in [0,1]$, $\lim_{n\to\infty} f_n(x) = f(x)$), then f is measurable.

Proof

(i) Let $f(x) = \sup\{f_n(x) : n \geq 1\}$. We must show $f : [0,1] \to \mathbb{R}^*$ is measurable.

However $f^{-1}((a,\infty]) = \{x \in [0,1] : \sup_n f_n(x) > a\}$

$$= \bigcup_{n=1}^{\infty} \{x \in [0,1] : f_n(x) > a\} \in \mathcal{M}([0,1]).$$

(As $\mathcal{M}([0,1])$ is closed under countable unions).

So f is measurable by criterion (iv) from Lemma 3.10.

The proof for $\inf_n f_n$ is left as an exercise.

Before we prove part (ii) we need to introduce limsup and liminf.

<u>Limsup and Liminf</u> (See Chapter 2.)

Let $(x_n)_{n\geq 1}$ be any sequence from $\mathbb{R}^*$.

We obtain another sequence $(\sup_{k\geq n} x_k)_{n\geq 1}$.

Here $\sup_{k\geq n} x_k := \sup\{x_k : k \geq n\}$ $(\in \mathbb{R}^*)$.

Notice that $(\sup_{k\geq n} x_k)_{n\geq 1}$ is a decreasing sequence (actually, by 2.12). Hence it necessarily has a limit in $\mathbb{R}^*$.

This limit is denoted $\limsup_{n\to\infty} x_n$.

Similarly $(\inf_{k\geq n} x_k)_{n\geq 1}$ is an increasing sequence, so has a limit in $\mathbb{R}^*$, denoted $\liminf_{n\to\infty} x_n$.

<u>Lemma 3.13'</u>

$\lim_{n\to\infty} x_n$ exists $\iff$ $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$. In the latter case, all three limits are equal.

<u>Proof.</u>

$\Rightarrow$: Suppose $\lim_{n\to\infty} x_n = l \in \mathbb{R}$. Then $x_n \in \mathbb{R}$ for sufficiently large n. Let $\varepsilon > 0$. Choose N so that $|x_n - l| < \varepsilon/2$ for all $n \geq N$. Then $l - \frac{\varepsilon}{2} < x_n < l + \varepsilon/2$ $\forall n \geq N$, so

$l - \frac{\varepsilon}{2} \leq \sup_{k\geq N} x_k \leq l + \frac{\varepsilon}{2}$ and $l - \frac{\varepsilon}{2} \leq \inf_{k\geq N} x_k \leq l + \frac{\varepsilon}{2}$ for all $n \geq N$.

Then $|\sup_{k\geq n} x_k - l| \leq \varepsilon$ for all $n \geq N$. Hence $\limsup_{n\to\infty} x_n = l$ and similarly for $\liminf_{n\to\infty} x_n$.

The cases that $\lim_{n\to\infty} x_n = \infty$ or $-\infty$ I leave as an exercise.