Chapter 2 Countability and Cantor Sets.

We use the following as our definition of countability. (It is not quite the usual one (from MATH10101, say) but works best for us.)

Definition

A set E is called countable if either $E = \emptyset$ (the empty set) or else there exists a surjection $f : \mathbb{N} \to E$, where $\mathbb{N} := \{1, 2, 3, \dots\}$ is the set of positive integers.

So a non-empty set is countable if its elements can be written in a (possibly repeating) list $x_1, x_2, \dots, x_n, \dots$ (Just take $x_n = f(n)$.) One can then weed out repeated elements to get a non-repeating list which will either terminate, $x_1, \dots, x_N$, in which case the set is finite (with N elements), or not, in which case we say that the set is countably infinite.

A set that is not countable is called uncountable.

By way of revision, I list some standard results about countability. The proofs are either sketched, or left as exercises.

Proposition 2.1

(i) If E is countable and $f : E \to F$ is a surjection, then F is countable.

(ii) Any subset of a countable set is countable.

Proof. Exercise.

Proposition 2.2

$\mathbb{Z}$ is countable.

Proof.

Define $f : \mathbb{N} \to \mathbb{Z}$ by $f(n) = \begin{cases} m & \text{if } n = 2m \\ 1-m & \text{if } n = 2m-1. \end{cases}$

Then f is surjective. $\square$

Proposition 2.3

If E and F are countable then (i) $E \cup F$ is countable, and (ii) $E \times F$ is countable.

Proof.

If one of E, F are empty, both (i) and (ii) are trivial. So assume that we have surjections $f : \mathbb{N} \to E$, $g : \mathbb{N} \to F$.

(i) Define $h : \mathbb{N} \to E \cup F$ by $h(n) = \begin{cases} f(n/2) & \text{if } n \text{ even}, \\ g\left(\frac{n+1}{2}\right) & \text{if } n \text{ odd}. \end{cases}$

Easy to show that $h : \mathbb{N} \to E \cup F$ is surjective. $\square$

(ii) Easy fact: every $n \in \mathbb{N}$ can be written uniquely in the form $2^{a-1}(2b-1)$ for some $a, b \in \mathbb{N}$.

Then $H(n) = (a, b)$ gives a surjection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$ (actually, a bijection). So $\mathbb{N} \times \mathbb{N}$ is countable.

So by 2.1 (i) it is sufficient to find a surjection $h : \mathbb{N} \times \mathbb{N} \to E \times F$. Just take $h((m,n)) = (f(m), g(n))$. $\square$

Proposition 2.4

$\mathbb{Q}$ is countable.

Proof.

$\mathbb{Z} \times \mathbb{Z}$ is countable by 2.2 and 2.3 (ii). Define $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ by $h((m,n)) = \begin{cases} 1 & \text{if } n = 0 \\ m/n & \text{if } n \neq 0. \end{cases}$

Clearly h is surjective, so $\mathbb{Q}$ is countable by 2.1 (i). $\square$

Proposition 2.5

A countable union of countable sets is countable. I.e. if, for each $n \in \mathbb{N}$, E_n is a countable set, then their union $\bigcup_{n=1}^{\infty} E_n$ is also a countable set.

Proof.
We may assume that for all $n \in \mathbb{N}$, $E_n \neq \emptyset$, so let $f_n : \mathbb{N} \to E_n$ be a surjection. Define $f : \mathbb{N} \times \mathbb{N} \to \bigcup_{n=1}^{\infty} E_n$ by $f((n,m)) = f_n(m)$. Clearly f is surjective. Hence $\bigcup_{n=1}^{\infty} E_n$ is countable by 2.1 (i) (and 2.3 (ii)). $\square$

Proposition 2.6

$\mathbb{R}$ is uncountable.

Proof.
Suppose not. Let $f : \mathbb{N} \to \mathbb{R}$ be a surjection. For each $n \in \mathbb{N}$, define $a_n \in \{0,1\}$ by

$$a_n = \begin{cases} 5 & \text{if the } n\text{'th decimal digit of } f(n) \text{ is not } 5; \\ 6 & \text{if the } n\text{'th decimal digit of } f(n) \text{ is } 5. \end{cases}$$

Let $\alpha = 0 \cdot a_1 a_2 a_3 \cdots$ (in decimal notation).
Since f is surjective, there exists $n_0 \in \mathbb{N}$ such that $f(n_0) = \alpha$. So the n_0'th decimal digit of $f(n_0)$ is a_{n_0} (exercise - it follows from ex. 2). But this contradicts the definition of a_{n_0}. $\square$

The Middle Third Cantor Set

We now describe a particular set $C \subseteq [0,1]$, called the Middle Third Cantor set which we return to later. Its arithmetic description in terms of ternary expansions of real numbers is very easy.

Recall that every $\alpha \in [0,1]$ can be written (not necessarily uniquely) in the form

$$\alpha = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \quad \text{---} (*)$$

where each a_n is either 0, 1 or 2. (Note that $\left|\frac{a_n}{3^n}\right| \leq 2 \cdot 3^{-n}$ so the series certainly converges by the Comparison Test.)

Then C is defined as the set of all those $\alpha \in [0,1]$ that can be written in the form $(*)$ with each a_n being either 0 or 2. In fact, the representation is then unique :—

<u>Lemma 2.7</u>

Let $a_n, b_n \in \{0,2\}$ for all $n \in \mathbb{N}$ and let $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ and $y = \sum_{n=1}^{\infty} \frac{b_n}{3^n}$. Suppose that for some m we have $a_m \neq b_m$.

Then $x \neq y$.

<u>Proof.</u>

Then set $\{n \in \mathbb{N} : a_n \neq b_n\}$ is non-empty by hypothesis. Let M be its least element. Then $a_n = b_n$ for $n < M$ and $|a_M - b_M| = 2$.

Now $\quad x - y = (a_M - b_M)3^{-M} + \sum_{n=M+1}^{\infty} (a_n - b_n) 3^{-n}$.

Recall that $|\alpha + \beta| \geq |\alpha| - |\beta|$, so

$$|x-y| \geq |a_M - b_M| 3^{-M} - \left| \sum_{n=M+1}^{\infty} (a_n - b_n) 3^{-n} \right|$$

$$\geq 2 \cdot 3^{-M} - \sum_{n=M+1}^{\infty} |a_n - b_n| \cdot 3^{-n}$$

$$\geq 2 \cdot 3^{-M} - \sum_{n=M+1}^{\infty} 2 \cdot 3^{-n} = 2 \cdot 3^{-M} - 3^{-(M+1)} \cdot 2 \cdot \sum_{n=0}^{\infty} 3^{-n}$$

$$= 2 \cdot 3^{-M} - 3^{-(M+1)} \cdot 2 \cdot \frac{1}{1 - 1/3}$$

$$= 2 \cdot 3^{-M} - 3^{-M}$$
$$= 3^{-M} > 0 .$$

$\therefore\ x \neq y$ as required. $\square$

<u>Corollary 2.8</u>

The formula $\sum_{n=1}^{\infty} a_n 3^{-n} \longmapsto (a_n)_{n=1}^{\infty}$

gives a well-defined bijective function between C and the set S of all sequences $(a_n)_{n=1}^{\infty}$, where $a_n \in \{0,2\}$ for all $n \in \mathbb{N}$. $\square$

<u>Theorem 2.9</u>

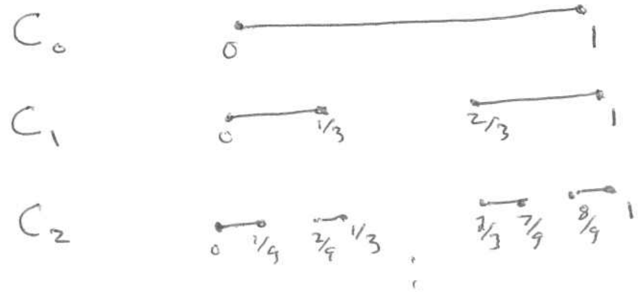
The Middle Third Cantor set C is uncountable.

<u>Proof</u>

Obviously S is bijective with $\{0,1\}^{\mathbb{N}}$ – see exercise 3, from which the Theorem now follows. $\square$

<u>Geometric description of C</u> : Those α (in $(*)$) with $a_1 = 1$ are precisely those $\alpha \in [0,1]$ with $\alpha \in (1/3, 2/3)$. So let $C_0 = [0,1]$, $\quad C_1 = C_0 \setminus (1/3, 2/3) = [0, \frac{1}{3}] \cup [2/3, 1]$. Then $C \subseteq C_1$. Those $\alpha \in C_1$ with $a_2 = 1$ (in $(*)$) are precisely those $\alpha \in C_1$ with $\alpha \in (\frac{1}{9}, 2/9)$ or $\alpha \in (7/9, 8/9)$. So let $C_2 = C_1 \setminus ((1/9, 2/9) \cup (7/9, 8/9))$

$$= [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$



Continue in this way: so C_n consists of 2^n disjoint closed intervals formed by removing the middle (open) thirds of the intervals making up C_{n-1}.

Then $C = \bigcap_{n=1}^{\infty} C_n$.

<u>Revision of sups and infs</u>.

Recall that if $\emptyset \neq E \subseteq \mathbb{R}$ and $m \in \mathbb{R}$, then we say that m is an <u>upper bound</u> for E if for all $x \in E$, $x \leq m$.
Similarly, is a <u>lower bound</u> for E if for all $x \in E$, $l \leq x$.
We write $\mathcal{U}(E)$ and $\mathcal{L}(E)$ for the set of upper and lower bounds for E, respectively:

$$\mathcal{U}(E) := \{m \in \mathbb{R} : \forall x \in E,\ x \leq m\}$$
$$\mathcal{L}(E) := \{l \in \mathbb{R} : \forall x \in E,\ l \leq x\}.$$

If $\mathcal{U}(E) \neq \emptyset$ (respectively $\mathcal{L}(E) \neq \emptyset$) we say that E is <u>bounded above</u> (respectively <u>bounded below</u>). If $\mathcal{U}(E) = \emptyset$ (resp. $\mathcal{L}(E) = \emptyset$) we say that E is <u>unbounded above</u> (resp. <u>unbounded below</u>).

Recall the Completeness Property:

(1) If E is bounded above then $\mathcal{U}(E)$ has a smallest member, called the supremum of E, and written $\sup E$.

(2) If E is bounded below then $\mathcal{L}(E)$ has a greatest member, called the infimum of E, and written $\inf E$.

In the case that E is unbounded above, we write $\sup E = +\infty$.
In the case that E is unbounded below, we write $\sup E = -\infty$.

<u>Proposition 2.10</u>

Let $E \subseteq \mathbb{R}$ be bounded above. Then

$$m = \sup E \iff m \in \mathcal{U}(E) \text{ and } \forall \varepsilon > 0\ \exists x \in E,\ m - \varepsilon < x$$

<u>Proof</u>. Exercise. $\square$

Proposition 2.11

Let $E \subseteq \mathbb{R}$ be bounded below. Then

$$l = \inf E \iff l \in \mathcal{L}(E) \text{ and } \forall \varepsilon > 0\ \exists x \in E,\ x < l + \varepsilon.$$

Proof. Exercise. $\square$

Proposition 2.12

Suppose that $\emptyset \neq E \subseteq F \subseteq \mathbb{R}$. Then $\sup E \leq \sup F$ and $\inf F \leq \inf E$.

Proof.

The printed notes do the argument for sup, so I'll do the argument for inf.

If $\inf F = -\infty$ the inequality is trivial. So assume F is bounded below, i.e. $\mathcal{L}(F) \neq \emptyset$. Let $l \in \mathcal{L}(F)$. Then $\forall x \in F$, $l \leq x$. So (as $E \subseteq F$) we have $\forall x \in E$, $l \leq x$ and hence $l \in \mathcal{L}(E)$. We have shown $\mathcal{L}(F) \subseteq \mathcal{L}(E)$. Let $l_F = \inf F$, $l_E = \inf E$. Since $l_F \in \mathcal{L}(F)$, we have $l_F \in \mathcal{L}(E)$. But l_E is the largest element of $\mathcal{L}(E)$, so $l_F \leq l_E$ as required. $\square$

Remarks on Example Sheet 2

You are now able to attempt questions 1, 2, 3 and 5. Question 4 concerns the section in the printed notes on lim sup and lim inf which I'll cover later when it's needed.