$\lim_{n\to\infty} g_n(x) = g(x)$, for μ-a.e. x.

Also,
$$|g(x)| = \lim_{n\to\infty} |g_n(x)| \leq \lim_{n\to\infty} \sum_{i=0}^{n-1} |g_{i+1}(x) - g_i(x)| = \lim_{n\to\infty} |h_n(x)| = h(x), \text{ for } \mu\text{-a.e. } x,$$

and hence $|g(x)|^2 \leq |h(x)|^2$ for μ-a.e. x and so $|g|^2$ is integrable, giving $g \in L^2([-\pi,\pi],\mu,\mathbb{R})$.

We also have that $|g(x) - g_n(x)|^2 \leq (|g(x)| + |g_n(x)|)^2 \leq (2h(x))^2$.

Since $\lim_{n\to\infty} |g(x) - g_n(x)|^2 = 0$ for μ-a.e. x, the Dominated Convergence Theorem tells us that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |g - g_n|^2 \, d\mu = 0$$

which implies that $\lim_{n\to\infty} \|g - g_n\|_2 = 0$.

Now, finally, let $\varepsilon > 0$.

Choose i so large that $\|g - g_i\|_2 < \varepsilon/2$ and $2^{-i} < \varepsilon/2$.

Since $g_i = f_{N_i}$ we have for all $n \geq N_i$:-

$$\|g - f_n\|_2 \leq \|g - g_i\|_2 + \|g_i - f_n\|_2 \quad \text{(as } \|\cdot\|_2 \text{ is a norm)}$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $\lim_{n\to\infty} \|g - f_n\|_2 = 0$, as required. $\square$

Orthogonality

Definition

Let V be a vector space with an inner product $\langle \cdot,\cdot \rangle$ and associated norm $\|v\| = \langle v,v\rangle^{1/2}$ (for $v \in V$). We say that a collection of vectors $\{v_n\}_{n\in S}$ $(S \subseteq \mathbb{Z})$ is <u>orthogonal</u> if $\langle v_n, v_m\rangle = 0$ whenever $n \neq m$, and <u>orthonormal</u> if, in addition, $\|v_n\| = 1$ for all $n \in S$.

Exercise

Let $\{v_1, v_2\}$ be orthogonal in $\mathbb{R}^2$ (for $\langle \cdot, \cdot \rangle$ given in example on p49). Then the vectors v_1, v_2 are at right angles to each other.

Lemma 4.10

Let $\{v_k\}_{k=1}^n$ be a finite orthogonal family in V and let $c_1, \ldots, c_n \in \mathbb{R}$. Then

$$\left\| \sum_{k=1}^n c_k v_k \right\|^2 = \sum_{k=1}^n c_k^2 \cdot \|v_k\|^2 .$$

Proof.

Using linearity of $\langle \cdot, \cdot \rangle$ and orthogonality

$$\| c_1 v_1 + c_2 v_2 \|^2 = \langle c_1 v_1 + c_2 v_2, c_1 v_1 + c_2 v_2 \rangle = c_1^2 \langle v_1, v_2 \rangle + 2 c_1 c_2 \langle v_1, v_2 \rangle + c_2^2 \langle v_2, v_2 \rangle$$

$$= c_1^2 \|v_1\|^2 + c_2^2 \|v_2\|^2 .$$

□

Now use induction on n.

Lemma 4.11

Let $\{v_k\}_{k=1}^n$ be a finite orthonormal family in V and let $w \in V$. Then the minimum value of

$$\left\| w - \sum_{k=1}^n c_k v_k \right\|$$

over all choices of $c_1, \ldots, c_n \in \mathbb{R}$ occurs when $c_k = \langle w, v_k \rangle$ for $k = 1, \ldots, n$.

Proof.

Let $c_1, \ldots, c_n \in \mathbb{R}$. Let $a_k = \langle w, v_k \rangle$ for $k = 1, \ldots, n$.

Let $u := \sum_{k=1}^n a_k v_k$, $v = \sum_{k=1}^n c_k v_k$.

By 4.10 $\|u\|^2 = \sum_{k=1}^n a_k^2$, $\|v\|^2 = \sum_{k=1}^n c_k^2$. We want $\min \|w - v\|$.

Also $\langle w, v \rangle = \left\langle w, \sum_{k=1}^n c_k v_k \right\rangle = \sum_{k=1}^n c_k \langle w, v_k \rangle = \sum_{k=1}^n c_k a_k$.

Now $\|w - v\|^2 = \langle w - v, w - v \rangle = \|w\|^2 - 2 \langle w, v \rangle + \|v\|^2$

$$= \|w\|^2 - 2 \sum_{k=1}^n c_k a_k + \sum_{k=1}^n c_k^2$$

$$= \|w\|^2 - \sum_{k=1}^n a_k^2 + \sum_{k=1}^n (a_k - c_k)^2$$

$$= \|w\|^2 - \|u\|^2 + \sum_{k=1}^n (a_k - c_k)^2$$

$\therefore$ $\|w - v\|^2 \geq \|w\|^2 - \|u\|^2$ for all choices of $c_1, \ldots, c_n$ with equality if and only if $c_k = a_k = \langle w, v_k \rangle$ for $k = 1, \dots, n$.

(In fact, setting $c_k = a_k$ for $k = 1, \dots, n$, we see that $\|w\|^2 - \|u\|^2$, which is independent of $c_1, \dots, c_n$, is non-negative and the minimum value of $\|w - v\|$ is $(\|w\|^2 - \|u\|^2)^{1/2}$.)

$\square$

__Definition__

Suppose that V is a Hilbert space and that $\{v_n\}_{n\in\mathbb{Z}}$ is an orthonormal set of vectors in V. Then $\{v_n\}_{n\in\mathbb{Z}}$ is called a __(Schauder) basis for $V__ if for all $w \in V$, there exists a sequence $\{c_n\}_{n\in\mathbb{Z}}$ of real numbers such that

$$w = \sum_{n\in\mathbb{Z}} c_n v_n,$$

i.e.
$$w = \lim_{n\to\infty} \sum_{k=-n}^{n} c_k v_k$$

i.e.
$$\lim_{n\to\infty} \|w - \sum_{k=-n}^{n} c_k v_k\| = 0.$$

__Remark__

In the above, the sequence $\{c_n\}_{n\in\mathbb{Z}}$ is necessarily unique. For if $\{d_n\}_{n\in\mathbb{Z}}$ were another sequence such that

$$\lim_{n\to\infty} \|w - \sum_{k=-n}^{n} d_k v_k\| = 0, \quad \text{then}$$

$$\lim_{n\to\infty} \|\sum_{k=-n}^{n} d_k v_k - \sum_{k=-n}^{n} c_k v_k\| = 0$$

i.e.
$$\lim_{n\to\infty} \|\sum_{k=-n}^{n} (d_k - c_k) v_k\| = 0 \text{ --- } (*)$$

Fix any $k_0 \in \mathbb{Z}$. By 4.10 we have, for $n \geq |k_0|$,

$$\|\sum_{k=-n}^{n} (d_k - c_k) v_k\| = \left(\sum_{k=-n}^{n} (d_k - c_k)^2 \|v_k\|^2\right)^{1/2}$$

$$= \left(\sum_{k=-n}^{n} (d_k - c_k)^2\right)^{1/2} \quad (\text{as } \|v_k\| = 1 \ \forall k \in \mathbb{Z})$$

$$\geq |d_{k_0} - c_{k_0}|$$

Taking limits as $n \to \infty$, we see by $(*)$ that $d_{k_0} = c_{k_0}$. □

It now follows that $c_k = \langle w, v_k \rangle$ for all $k \in \mathbb{Z}$.

Indeed, for any $n \geq 1$,

$$\Big\| w - \sum_{k=-n}^{n} c_k v_k \Big\| \geq \Big\| w - \sum_{k=-n}^{n} \langle w, v_k \rangle v_k \Big\| \quad \text{by 4.11,}$$

so letting $n \to \infty$ we see that $\lim_{n\to\infty} \big\| w - \sum_{k=-n}^{n} \langle w, v_k \rangle v_k \big\| = 0$,

so by the uniqueness, $c_k = \langle w, v_k \rangle$ for each $k \in \mathbb{Z}$. □

Fourier Series

Consider the set

$$\mathcal{F} = \left\{ \frac{1}{\sqrt{2}},\ \cos(nx),\ \sin(nx) : n \geq 1 \right\}$$

of functions with domain $[-\pi, \pi]$

It is convenient to write, for $n \in \mathbb{Z}$,

$$\phi_n(x) = \begin{cases} \cos nx & \text{if } n > 0 \\ 1/\sqrt{2} & \text{if } n = 0 \\ \sin(-nx) & \text{if } n < 0. \end{cases}$$

Clearly ϕ_n^2 is integrable (being continuous, and hence even Riemann integrable) and so $\phi_n \in L^2([-\pi,\pi], \mu, \mathbb{R})$ for each $n \in \mathbb{Z}$.

We aim to show that $\mathcal{F} = \{\phi_n : n \in \mathbb{Z}\}$ is a basis for $L^2([-\pi,\pi], \mu, \mathbb{R})$, and the $L^2([-\pi,\pi], \mu, \mathbb{R})$-representation

$$f = \sum_{n=-\infty}^{\infty} \langle f, \phi_n \rangle \phi_n$$

is precisely the Fourier series representation of $f \in L^2([-\pi,\pi], \mu, \mathbb{R})$.

However, it must be pointed out that the convergence here is <u>not pointwise</u>, but in the L^2-norm.

Now for $n>0$, we have $\langle f, \phi_n\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,d\mu = a_n$, and

$\langle f, \phi_{-n}\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,d\mu = b_n$, and for $n=0$,

$\langle f, \phi_0\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(x)}{\sqrt{2}}\,d\mu$, so $\langle f, \phi_0\rangle\phi_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,d\mu = \frac{a_0}{2}$

so these are the usual Fourier coefficients. But note that we only assume Lebesgue integrability of f^2 (and not Riemann integrability of f).

It is also easy to see by calculation (of Riemann integrals) that $\mathcal{F}$ is an orthonormal set:-

$\langle \phi_0, \phi_0\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\,dx = \frac{1}{2\pi}\left[x\right]_{-\pi}^{\pi} = 1;$

for $n>0$, $\langle \phi_n, \phi_n\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 nx\,dx = 1;$

for $n<0$, $\langle \phi_n, \phi_n\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(-nx)\,dx = 1;$

for $n\neq 0$, $\langle \phi_n, \phi_0\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos(nx)}{\sqrt{2}}\,dx$ or $\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin nx}{\sqrt{2}}\,dx$, both $=0$.

So it remains to show that for $n\neq 0, m\neq 0, n\neq m$ (in $\mathbb{Z}$), $\frac{1}{\pi}\int_{-\pi}^{\pi} \phi_n(x)\,\phi_m(x)\,dx = 0$. The easiest way to do this is probably to use the formulas $\cos(nx) = \frac{e^{inx}+e^{-inx}}{2}$, $\sin(nx) = \frac{e^{inx}-e^{-inx}}{2i}$

and $\frac{1}{\pi}\int_{-\pi}^{\pi} e^{ikx}\,d\mu = \begin{cases} 0 & \text{if } k\neq 0 \\ 2 & \text{if } k=0 \end{cases}$.

So it remains to show:

<u>Theorem 4.13</u> (Riesz - Fischer Theorem) (1907)

Let $f \in L^2([-\pi,\pi], \mathcal{M}, \mathbb{R})$ and set

$$S_n(f,x) := \sum_{k=-n}^{n} \langle f, \phi_k\rangle \phi_k(x) \quad (\text{for } x\in[-\pi,\pi]).$$

Then $S_n(f,\cdot)$ converges to f in the $\|\cdot\|_2$-norm, i.e.

$$\| S_n(f,\cdot) - f \|_2 = \left(\frac{1}{\pi} \int_{-\pi}^{\pi} | S_n(f,\cdot) - f |^2 \, d\mu \right)^{1/2} \to 0, \text{ as } n \to +\infty.$$

Proof.

We first define, for any bounded function $f : [-\pi, \pi] \to \mathbb{R}$, the ∞-norm, $\|f\|_\infty := \sup_{x \in [-\pi,\pi]} |f(x)|$. Then for $f \in L^2([-\pi,\pi], \mu, \mathbb{R})$ and f bounded, we have

$$\|f\|_2 = \left(\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, d\mu \right)^{1/2} \leq \left(\frac{1}{\pi} \int_{-\pi}^{\pi} \|f\|_\infty^2 \, d\mu \right)^{1/2}$$

$$= \left(\frac{\|f\|_\infty^2}{\pi} \cdot \int_{-\pi}^{\pi} 1 \, d\mu \right)^{1/2}$$

$$= \sqrt{2} \cdot \|f\|_\infty \quad \text{---- } (*)$$

Now recall Fejér's Theorem (1.3): if $g : [-\pi, \pi] \to \mathbb{R}$ is continuous and $g(-\pi) = g(\pi)$ (so g extends by periodicity to all of $\mathbb{R}$), then

$$\lim_{n \to \infty} \| \sigma_n(g,\cdot) - g \|_\infty = 0, \quad \text{where, for } n \geq 1 \text{ and } x \in [-\pi, \pi],$$

$$\sigma_n(g,x) := \frac{1}{n} \left(S_0(f,x) + S_1(f,x) + \cdots + S_{n-1}(f,x) \right)$$

$$\underset{(\text{exercise})}{=} \sum_{k=-(n-1)}^{n-1} \frac{n - |k|}{n} \langle f, \phi_k \rangle \phi_k(x) \quad \text{---} \quad (**)$$

Now let $f \in L^2([-\pi,\pi], \mu, \mathbb{R})$ and let $\varepsilon > 0$ be given. We must find $N \geq 1$ such that for all $n \geq N$, $\|f - S_n(f,\cdot)\|_2 < \varepsilon$.

Now by 4.6 we may choose a continuous $g : [-\pi,\pi] \to \mathbb{R}$ with $g(-\pi) = g(\pi)$ such that $\|f - g\|_2 < \varepsilon/2$.

By Fejér's Theorem (1.3), find N such that

$$\forall n \geq N, \qquad \| \sigma_n(g,\cdot) - g \|_\infty < \frac{\varepsilon}{2\sqrt{2}}.$$

Hence by $(*)$ (since $\sigma_n(g,\cdot) - g$ is continuous on $[-\pi,\pi]$ and hence is in $L^2([-\pi,\pi], \mu, \mathbb{R})$), $\| \sigma_n(g,\cdot) - g \|_2 < \frac{\varepsilon}{2}$.

So for $n \geq N$, $\| f - \sigma_n(g,\cdot) \|_2 \leq \| f - g \|_2 + \| g - \sigma_n(g,\cdot) \|_2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Now $\sigma_n(g,x)$ has the form $\sum_{k=-(n-1)}^{n-1} d_k \phi_k(x)$ for some $d_{-(n-1)}, \ldots, d_{n-1} \in \mathbb{R}$.

(This follows from $(**)$.) since the $S_k(f,x)$'s have this form by their definition.)

So $\|f - \sum_{k=-(n-1)}^{n-1} d_k \cdot \phi_k\|_2 < \varepsilon$.

So using 4.11 we get, $\forall n \geq N$,

$$\|f - \sum_{k=-(n-1)}^{n-1} \langle f, \phi_k \rangle \phi_k\|_2 \leq \|f - \sum_{k=-(n-1)}^{n-1} d_k \cdot \phi_k\|_2 < \varepsilon,$$

i.e. $\|f - S_n(f,\cdot)\|_2 < \varepsilon$, $\forall n \geq N+1$, as required. $\square$

This still says nothing about pointwise convergence of $S_n(f,x)$ to $f(x)$, nor about f that are merely integrable. In fact, there exist examples of integrable functions $f : [-\pi,\pi] \to \mathbb{R}$ such that $S_n(f,x)$ converges for <u>no</u> $x \in [-\pi,\pi]$ [Kolmogorov, 1923]. However, for L^2-functions we have the following:

<u>Theorem 4.15</u> (Carleson, 1966).

Let $f \in L^2([-\pi,\pi], \mu, \mathbb{R})$. Then $S_n(f,x)$ converges to $f(x)$, for μ-a.e. $x \in [-\pi,\pi]$, as $n \to +\infty$.