Lemma 4.2

Let $f, g \in L^2([-\pi,\pi], \mu, \mathbb{R})$. Then

(i) Hölder Inequality:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |fg| \, d\mu \le \|f\|_2 \cdot \|g\|_2$$

with equality if and only if for some $c \in \mathbb{R}$, $|f| = c|g|$, μ-a.e., or $\|f\|_2 \cdot \|g\|_2 = 0$.

(ii) Cauchy-Schwarz Inequality:

$$\left| \frac{1}{\pi}\int_{-\pi}^{\pi} fg \, d\mu \right| \le \|f\|_2 \, \|g\|_2$$

with equality as in (i).

Proof.

(i) Obviously if $\|f\|_2 = 0$ or $\|g\|_2 = 0$ then $f = 0$ μ-a.e. or $g = 0$ μ-a.e., so $f \cdot g = 0$ μ-a.e. and we have equality. So let $A = \|f\|_2$, $B = \|g\|_2$ and assume that $A \neq 0$ and $B \neq 0$ (so $A > 0$, $B > 0$).

Now apply Lemma 4.1 to the functions $\frac{|f|}{A}$, $\frac{|g|}{A}$:-

$$\frac{2}{\pi}\int_{-\pi}^{\pi} \frac{|f|}{A} \cdot \frac{|g|}{B} \, d\mu \le \left\| \frac{|f|}{A} \right\|_2^2 + \left\| \frac{|g|}{B} \right\|_2^2 = \frac{1}{A^2} \cdot \|f\|_2^2 + \frac{1}{B^2} \cdot \|g\|_2^2 = 1 + 1 = 2$$

$$\therefore \frac{1}{\pi}\int_{-\pi}^{\pi} |fg| \, d\mu \le 1 \cdot A \cdot B = \|f\|_2 \cdot \|g\|_2 \quad \text{as required.} \ \square$$

Further, we have equality if and only if $\frac{|f|}{A} = \frac{|g|}{B}$ μ-a.e., i.e. if and only if $|f| = c \cdot |g|$ μ-a.e., where $c = A/B$. $\square$

(ii) The inequality follows immediately from Hölder (and 3.19 (i)), but the conditions for equality ~~do not~~. Instead we argue as follows.

We may, as above assume that $\|f\|_2 \neq 0$ and $\|g\|_2 \neq 0$.

Let $t = \frac{1}{\|f\|_2^2} \cdot \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} fg \, d\mu$ $(\in \mathbb{R})$ and apply 4.1 to the functions tf and g. This gives

$$\frac{2}{\pi^2}\cdot\frac{1}{\|f\|_2^2}\left(\int_{-\pi}^{\pi} fg\,d\mu\right)^2 \le \left(\underbrace{\frac{1}{\|f\|_2^2}\cdot\frac{1}{\pi}\int_{-\pi}^{\pi} fg\,d\mu}_{t}\right)^2\cdot\|f\|_2^2 + \|g\|_2^2$$

and multiplying both sides by

$$= \frac{1}{\|f\|_2^2}\cdot\frac{1}{\pi^2}\left(\int_{-\pi}^{\pi} fg\,d\mu\right)^2 + \|g\|_2^2$$

$\therefore$, taking first term on RHS over to the LHS, and multiplying by $\|f\|_2^2$ we get (upon taking +ve square roots):

$$\left|\frac{1}{\pi}\int_{-\pi}^{\pi} fg\,d\mu\right| \le \|f\|_2\cdot\|g\|_2 \;,$$ with

equality if and only if $tf = g$ μ-a.e.. This reduces to (exercise) iff $f = cg$ μ-a.e. for some $c \in \mathbb{R}$ (with $c = \pm \|f\|_2 / \|g\|_2$ in fact). $\square$

<u>Lemma 4.3</u> (Minkowski's Inequality)

If $f, g \in L^2([-\pi,\pi],\mu,\mathbb{R})$ then so is $f+g$ and

$$\|f+g\|_2 \le \|f\|_2 + \|g\|_2 .$$ (Minkowski

Proof.

We know that $f+g$ is measurable (3.14). Also $\int_{-\pi}^{\pi}(f+g)^2\,d\mu$ is well-defined (as $(f+g)^2 \ge 0$) and

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f+g)^2\,d\mu = \frac{1}{\pi}\int_{-\pi}^{\pi}(f^2 + 2fg + g^2)\,d\mu$$

$$\le \frac{1}{\pi}\int_{-\pi}^{\pi}(f^2 + 2|fg| + g^2)\,d\mu$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f^2\,d\mu + \frac{2}{\pi}\int_{-\pi}^{\pi}|fg|\,d\mu + \frac{1}{\pi}\int_{-\pi}^{\pi} g^2\,d\mu$$

$$\le \|f\|_2^2 + 2\|f\|_2\|g\|_2 + \|g\|_2^2 \quad \text{(Hölder)}$$

$$= (\|f\|_2 + \|g\|_2)^2 < \infty$$

Hence $f+g \in L^2([-\pi,\pi],\mu,\mathbb{R})$ and taking square roots gives the result $\square$

<u>Corollary</u>

$L^2([-\pi,\pi],\mu,\mathbb{R})$ is a vector space over $\mathbb{R}$.

<u>Proof.</u>

It is trivial that $f \in L^2([-\pi,\pi],\mu,\mathbb{R}) \Rightarrow cf \in L^2([-\pi,\pi],\mu,\mathbb{R})$ (for $c \in \mathbb{R}$; note that $\|cf\|_2 = |c| \cdot \|f\|_2$). And by the above lemma, if $f, g \in L^2([-\pi,\pi],\mu,\mathbb{R})$ then so is $f+g \in L^2([-\pi,\pi],\mu,\mathbb{R})$.

$\square$

<u>Remark.</u>

Strictly speaking we should show $+$ is well-defined on $L^2([-\pi,\pi],\mu,\mathbb{R})$, i.e. if $f_0 = f_1$ μ-a.e and $g_0 = g_1$ μ-a.e. then $f_0 + g_0 = f_1 + g_1$ μ-a.e. I'll leave this to you.

<u>Theorem 4.4</u>

$\|f\|_2$ is a norm on the vector space $L^2([-\pi,\pi],\mu,\mathbb{R})$.

~~with~~ The associated metric is $d(f,g) := \|f-g\|_2$.

<u>Proof.</u>

(A) $\|f\|_2 = +\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f^2\, d\mu\right)^{1/2} \geq 0$ and we saw that equality holds iff $f = 0$ in $L^2([-\pi,\pi],\mu,\mathbb{R})$ in the proof of 4.1.

(B) $\|cf\|_2 = +\left(\frac{1}{\pi}\int_{-\pi}^{\pi} c^2 f^2\, d\mu\right)^{1/2} = |c| \cdot \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f^2\, d\mu\right)^{1/2} = |c| \cdot \|f\|_2$.

$+^{ve}$ sq. root of c^2.

(C) $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$ by 4.3.

Easily follows that $d(\cdot,\cdot)$ is a metric on $L^2([-\pi,\pi],\mu,\mathbb{R})$.

$\square$

<u>Definition</u>

For $f, g \in L^2([-\pi,\pi],\mu,\mathbb{R})$, $\langle f,g\rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} fg\, d\mu$ (which is $\in \mathbb{R}$ by 4.1).

<u>Easy exercise</u> Check $p(1)$, $p(2)$ and $p(3)$ hold, so that $\langle\cdot,\cdot\rangle$ is an inner product on $L^2([-\pi,\pi],\mu,\mathbb{R})$ with associated norm $\|\cdot\|_2$.

We aim to show:

Theorem 4.7

$L^2([-\pi,\pi],\mu,\mathbb{R})$ is a complete metric space, and hence a Hilbert space.

We now state for later use, the following:

Theorem 4.5

Continuous functions are $\|\cdot\|_2$-dense in $L^2([-\pi,\pi],\mu,\mathbb{R})$. This means that for all $f \in L^2([-\pi,\pi],\mu,\mathbb{R})$ and all $\varepsilon > 0$, we can find a continuous function $g : [-\pi,\pi] \to \mathbb{R}$ such that $\|f-g\|_2 < \varepsilon$.

Proof.

Non examinable. $\square$

Corollary 4.6

Same as 4.5 except the g can be found with the extra property that $g(-\pi) = g(\pi)$.

Proof.

Exercise – see printed notes for a hint. $\square$

Proof of 4.7

Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $L^2([-\pi,\pi],\mu,\mathbb{R})$. So by definition, $\forall \varepsilon > 0$ there exists $N \geq 1$ such that

$$n, m \geq N \implies \|f_n - f_m\|_2 < \varepsilon.$$

Apply this with $\varepsilon = 2^{-i}$ (for each $i \geq 1$) to obtain positive integers N_i such that

$$n, m \geq N_i \implies \|f_n - f_m\|_2 < \frac{1}{2^i}.$$

Let $g_0 = 0$, $g_i = f_{N_i}$ for $i \geq 1$, so that $\forall i \geq 1$,

$$\|g_{i+1} - g_i\|_2 = \|f_{N_{i+1}} - f_{N_i}\| < \frac{1}{2^i}.$$

By the Comparison Test for series it follows that

$$\sum_{i=0}^{\infty} \|g_{i+1} - g_i\|_2 \quad \text{converges.}$$

Let $S \equiv$ the sum of this series.

Our aim is to show that the series $\sum_{i=0}^{\infty}(g_{i+1}(x)-g_i(x))$ converges for μ-a.e. x, to $g(x)$ say, and that g is in $L^2([-\pi,\pi],\mu,\mathbb{R})$, and that $\|f_n-g\|_2\to 0$ as $n\to\infty$, so that the sequence $(f_n)_{n\geq 1}$ converges to g in the L^2-norm.

To this end we first consider the sequence of functions $(h_n)_{n\geq 1}$ defined by

$$h_n(x) = \sum_{i=0}^{n-1} |g_{i+1}(x)-g_i(x)|.$$

This is an increasing sequence of non-negative, measurable functions so that $h := \lim_{n\to\infty} h_n$ exists and is a measurable function from $[-\pi,\pi]$ to $\mathbb{R}^*$.

We want to show that $h\in L^2([-\pi,\pi],\mu,\mathbb{R})$.

Now $\|h_n\|_2 \leq \sum_{i=0}^{n-1}\|g_{i+1}-g_i\|_2$ (since $\|\cdot\|_2$ is a norm)

$$\leq S,$$

so $$\int_{-\pi}^{\pi} h_n^2\, d\mu = \pi\|h_n\|^2 \leq \pi S^2.$$

Also, $(h_n^2)_{n\geq 1}$ is an increasing sequence of non-negative measurable functions converging pointwise to h^2, so by the Monotone Convergence Theorem, $\int_{-\pi}^{\pi} h^2\, d\mu = \lim_{n\to\infty}\int_{-\pi}^{\pi} h_n^2\, d\mu \leq \pi S^2$, so $h^2 = |h|^2$ is integrable and hence $h\in L^2([-\pi,\pi],\mu,\mathbb{R})$ as required.

Now since h^2 is integrable it is finite μ-a.e., and hence h is finite μ-a.e. Further, whenever $h(x)$ is finite, the series of real numbers

$$\sum_{i=0}^{\infty}(g_{i+1}(x)-g_i(x))$$

converges absolutely, and hence converges, and we denote its sum by $g(x)$. If $h(x)=+\infty$, we set $g(x)=0$.

Now $\sum_{i=0}^{n-1}(g_{i+1}(x)-g_i(x)) = g_n(x)-g_0(x) = g_n(x)$, so in fact

$$\lim_{n\to\infty} g_n(x) = g(x), \text{ for } \mu\text{-a.e. } x.$$

Also, $|g(x)| = \lim_{n\to\infty} |g_n(x)|$

$$\leq \lim_{n\to\infty} \sum_{i=0}^{n-1} |g_{i+1}(x) - g_i(x)|$$

$$= \lim_{n\to\infty} |h_n(x)| = h(x), \text{ for } \mu\text{-a.e. } x,$$

and hence $|g(x)|^2 \leq |h(x)|^2$ for μ-a.e. x and so $|g|^2$ is integrable, giving $g \in L^2([-\pi,\pi], \mu, \mathbb{R})$.

We also have that $|g(x) - g_n(x)|^2 \leq (|g(x)| + |g_n(x)|)^2 \leq (2h(x))^2$.

Since $\lim_{n\to\infty} |g(x) - g_n(x)|^2 = 0$ for μ-a.e. x, the Dominated Convergence Theorem tells us that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |g - g_n|^2 \, d\mu = 0$$

which implies that $\lim_{n\to\infty} \|g - g_n\|_2 = 0$.

Now, finally, let $\varepsilon > 0$.

Choose i so large that $\|g - g_i\|_2 < \varepsilon/2$ and $2^{-i} < \varepsilon/2$.

Since $g_i = f_{N_i}$ we have for all $n \geq N_i$:-

$$\|g - f_n\|_2 \leq \|g - g_i\|_2 + \|g_i - f_n\|_2 \quad \text{(as } \|\cdot\|_2 \text{ is a norm)}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $\lim_{n\to\infty} \|g - f_n\|_2 = 0$, as required. $\square$