Lemma 4.2
Let $s, q \in L^2([-\pi, \pi], \mu, \mathbb{R})$. Then

(i) Hölder Inequality:

\[ \frac{1}{\pi} \int_{-\pi}^{\pi} |f \cdot g| \, d\mu \leq \|f\|_1 \cdot \|g\|_q \]

with equality if and only if for some $c \in \mathbb{R}$, $|s| = c|q|$, $\mu$-a.e. or $\|f\|_1 \cdot \|g\|_q = 0$.

(ii) Cauchy-Schwarz Inequality:

\[ \left| \frac{1}{\pi} \int_{-\pi}^{\pi} f \cdot g \, d\mu \right| \leq \|f\|_2 \cdot \|g\|_2 \]

with equality if and only if $f \cdot g = c \cdot |q|$, $\mu$-a.e.

Proof:

(i) Obviously if $\|s\|_1 = 0$ or $\|q\|_2 = 0$ then $s = 0 \mu$-a.e. or $q = 0 \mu$-a.e. and we have equality. So let $A = \|s\|_2$, $B = \|q\|_2$, and $f \cdot g = 0 \mu$-a.e. and we have equality.

Now apply Lemma 4.1 to the functions \( \frac{s}{A} \) and \( \frac{q}{B} \):

\[ \frac{2}{\pi} \int_{-\pi}^{\pi} \left( \frac{|s|}{A} \cdot \frac{|q|}{B} \right) \, d\mu \leq \left\| \frac{|s|}{A} \right\|_2^2 + \left\| \frac{|q|}{B} \right\|_2^2 = \frac{1}{A^2} \cdot \|s\|_2^2 + \frac{1}{B^2} \cdot \|q\|_2^2 = 1 + 1 = 2 \]

as required.

Further, we have equality if and only if $\frac{|s|}{A} = \frac{|q|}{B}$, $\mu$-a.e., i.e.

\[ \text{if and only if } |s| = c|q|, \mu \text{-a.e., where } c = \frac{A}{B}. \]

(ii) The inequality follows immediately from Hölder (and 3.19 (i)), but the condition for equality does not hold. Instead we argue as follows.

We may, as above, assume that $\|s\|_2 \neq 0$ and $\|q\|_2 \neq 0$.

Let $\xi = \frac{1}{\|s\|_2^2} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} f \cdot g \, d\mu \in \mathbb{R}$ and apply 4.1 to the functions $\xi f$ and $g$. This gives
\[
\frac{2}{\pi^2} \cdot \frac{1}{\|y\|_2^2} \left( \int_{-\pi}^{\pi} f_g \, d\mu \right)^2 \leq \left( \frac{1}{\|y\|_2} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} f_g \, d\mu \right)^2 \|y\|_2^2 + \|y\|_2^2
\]

and multiplying both sides by \(\frac{1}{\|y\|_2^2} \cdot \frac{1}{\pi^2} \)

\[
= \frac{1}{\|y\|_2^2} \cdot \frac{1}{\pi^2} \left( \int_{-\pi}^{\pi} f_g \, d\mu \right)^2 + \|y\|_2^2
\]

i.e., taking first term on RHS over to the LHS, and multiplying by \(\|y\|_2^2\), we get (upon taking square root,)

\[
\left| \frac{1}{\pi} \int_{-\pi}^{\pi} f \, d\mu \right| \leq \|y\|_2 \cdot \|g\|_2 , \text{ with equality if and only if } f = g \text{ } \mu\text{-a.e. This reduces to equality if and only if } f = cg \text{ } \mu\text{-a.e. for some } c \in \mathbb{R} \text{ (with } c = \pm \|y\|_2 / \|g\|_2 \text{ in fact).}
\]

Lemma 4.3 (Minkowski's Inequality)

\[
\forall f, g \in L^2([-\pi, \pi], \mu, IR) \text{ then } \|f + g\|_2 \leq \|f\|_2 + \|g\|_2.
\]

Proof. \(\text{We know that } f + g \text{ is measurable (3.14). Also, } \int_{-\pi}^{\pi} (f + g)^2 \, d\mu \text{ is well-defined (as } (f + g)^2 \geq 0) \text{ and}
\)

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} (f + g)^2 \, d\mu = \frac{1}{\pi} \int_{-\pi}^{\pi} (f^2 + 2fg + g^2) \, d\mu
\]

\[
\leq \frac{1}{\pi} \int_{-\pi}^{\pi} (f^2 + 2|fg| + g^2) \, d\mu
\]

\[
= \frac{1}{\pi} \int_{-\pi}^{\pi} f^2 \, d\mu + \frac{2}{\pi} \int_{-\pi}^{\pi} |fg| \, d\mu + \frac{1}{\pi} \int_{-\pi}^{\pi} g^2 \, d\mu
\]

\[
\leq \|f\|_2^2 + 2\|f\|_2 \cdot \|g\|_2 + \|g\|_2^2 \text{ (H"older)}
\]

\[
= (\|f\|_2^2 + \|g\|_2^2)^2 < \infty
\]

Hence \(f + g \in L^2([-\pi, \pi], \mu, IR)\) and taking square roots gives the result. \(\square\)
Corollary

\[ L^2\left([-\pi, \pi], \mu, IR\right) \] is a vector space over \( IR \).

**Proof.**

It is trivial that \( f \in L^2\left([-\pi, \pi], \mu, IR\right) \Rightarrow cf \in L^2\left([-\pi, \pi], \mu, IR\right) \) (for \( c \in IR \); note that \( \|cf\|_2 = |c| \cdot \|f\|_2 \)). And by the above lemma, if \( f, g \in L^2\left([-\pi, \pi], \mu, IR\right) \) then so it \( f + g \in L^2\left([-\pi, \pi], \mu, IR\right) \).

**Remark.**

Strictly speaking, we should show \( \cdot + \cdot \) is well-defined on \( L^2\left([-\pi, \pi], \mu, IR\right) \), i.e., if \( f_0 = f_1 \mu-a.e. \) and \( g_0 = g_1 \mu-a.e. \) then \( f_0 + g_0 = f_1 + g_1, \mu-a.e. \). We'll leave this to you.

**Theorem 4.4.**

\( \|f\|_2 \) is a norm on the vector space \( L^2\left([-\pi, \pi], \mu, IR\right) \).

The associated metric \( d(f, g) := \|f - g\|_2 \).

**Proof.**

(a) \( \|f\|_2 = \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f^2 \, d\mu\right)^{1/2} \geq 0 \) and we saw that this equality holds

iff \( f = 0 \) in \( L^2\left([-\pi, \pi], \mu, IR\right) \) in the proof of 4.1.

(b) \( \|c f\|_2 = \left(\frac{1}{\pi} \int_{-\pi}^{\pi} c^2 f^2 \, d\mu\right)^{1/2} = |c| \cdot \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f^2 \, d\mu\right)^{1/2} = |c| \cdot \|f\|_2 \).

(c) \( \|f + g\|_2 \leq \|f\|_2 + \|g\|_2 \) by 4.3.

Easily follows that \( d(\cdot, \cdot) \) is a metric on \( L^2\left([-\pi, \pi], \mu, IR\right) \).

**Definition.**

For \( f, g \in L^2\left([-\pi, \pi], \mu, IR\right) \), \( \langle f, g \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f g \, d\mu \) (which is \( \in IR \) by 4.1).

**Example.** Check \( p(1), p(\pi), p(3) \) hold, so that \( \langle \cdot, \cdot \rangle \) is an inner product on \( L^2\left([-\pi, \pi], \mu, IR\right) \) with associated norm \( \|\cdot\|_2 \).
We aim to show:

**Theorem 4.7**

$L^2 ([-\pi, \pi], \mu, IR)$ is a complete metric space, and hence a Hilbert space.

We now state for later use, the following:

**Theorem 4.5**

Continuous functions are $L^1$-dense in $L^2 ([-\pi, \pi], \mu, IR)$. This means that for all $f \in L^2 ([-\pi, \pi], \mu, IR)$ and all $\varepsilon > 0$, we can find a continuous function $g : [-\pi, \pi] \to IR$ such that $\| f - g \|_2 < \varepsilon$.

**Proof:**

Non examinable.

**Corollary 4.6**

Same as 4.5 except the $g$ can be found with the extra property that $g(-\pi) = g(\pi)$.

**Proof:**

Exercise — see problem notes for a hint.

**Proof of 4.7**

Let $(f_n)_{n \geq 1}$ be a Cauchy sequence in $L^2 ([-\pi, \pi], \mu, IR)$. So by definition, for $\varepsilon > 0$ there exists $N > 1$ such that

$n, m \geq N \implies \| f_n - f_m \|_2 < \varepsilon$.

Apply this with $\varepsilon = 2^{-i}$ (for each $i \geq 1$) to obtain pairwise integers $N_i$ such that

$n, m \geq N_i \implies \| f_n - f_m \|_2 < \frac{1}{2^i}$.

Let $g_0 = 0$, $g_i = f_{N_i}$ for $i > 1$, so that $\forall i \geq 1$,

$\| g_{i+1} - g_i \|_2 = \| f_{N_{i+1}} - f_{N_i} \|_2 < \frac{1}{2^i}$.

By the Comparison Test for series it follows that

$\sum_{i=0}^{\infty} \| g_{i+1} - g_i \|_2$ converges.

Let $S = \text{the sum of this series}$.
Our aim is to show that the series \( \sum_{i=0}^{\infty} (g_{i+1}(x) - g_i(x)) \) converges for \( \mu \)-a.e. \( x \), \( \text{sgn}(x) \), and that \( g \) is in \( L^2([-\pi, \pi], \mu, \mathbb{R}) \), and that \( \| S_n - g \|_2 \to 0 \) as \( n \to \infty \), so that the sequence \( (f_n)_{n \geq 1} \) converges to \( g \) in the \( L^2 \)-norm.

To this end we first construct the sequence of functions \( (f_n)_{n \geq 1} \) defined by
\[
h_n(x) = \sum_{i=0}^{n-1} |g_{i+1}(x) - g_i(x)|.
\]

This is an increasing sequence of non-negative, measurable functions so that \( h = \lim_{n \to \infty} h_n \) exists and is a measurable function from \([-\pi, \pi]\) to \( \mathbb{R}^+ \).

We want to show that \( h \in L^2([-\pi, \pi], \mu, \mathbb{R}) \).

Now
\[
\| h_n \|_2 \leq \sum_{i=0}^{n-1} \| g_{i+1} - g_i \|_2 \quad \text{(since \( \| \cdot \|_2 \) is a norm)}
\]
\[
\leq S,
\]
so
\[
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} h_n^2 \, d\mu = \int_{-\pi}^{\pi} \| h_n \|_2^2 \leq \pi S^2.
\]

Also, \( (h_n^2)_{n \geq 1} \) is an increasing sequence of non-negative measurable functions converging pointwise to \( h^2 \), so by the Monotone Convergence Theorem,
\[
\int_{-\pi}^{\pi} h_n^2 \, d\mu = \lim_{n \to \infty} \int_{-\pi}^{\pi} h_n^2 \, d\mu \lesssim \pi S^2,
\]
so \( h^2 \) is integrable and hence \( h \in L^2([-\pi, \pi], \mu, \mathbb{R}) \) as required.

Now since \( h^2 \) is integrable it is finite \( \mu \)–a.e., and hence \( h \) is finite \( \mu \)–a.e. Further, whenever \( h(x) \) is finite, the series of real numbers
\[
\sum_{i=0}^{\infty} (g_{i+1}(x) - g_i(x))
\]
converges absolutely and hence converges, and we denote its sum by \( g(x) \). If \( h(x) = +\infty \), we set \( g(x) = 0 \).

Now
\[
\sum_{i=0}^{n-1} (g_{i+1}(x) - g_i(x)) = g_n(x) - g_0(x) = g_n(x),
\]
so in fact
\[
\lim_{n \to \infty} g_n(x) = g(x), \quad \text{for } \mu\text{-a.e. } x.
\]

Also, \[|g_n(x)| = \lim_{n \to \infty} |g_n(x)|\]

\[\leq \lim_{n \to \infty} \sum_{i=0}^{n-1} \left| g_{i,n}(x) - g_{i,n}(x) \right|\]

\[= \lim_{n \to \infty} \left| h_n(x) \right| = h(x), \quad \text{for } \mu\text{-a.e. } x,
\]

and hence \[|g(x)|^2 \leq |h(x)|^2 \quad \text{for } \mu\text{-a.e. } x \quad \text{and } \quad \text{no } |g|^2 \quad \text{is integrable, giving } \quad g \in L^2([-\pi, \pi], \mu, \mathbb{R}).\]

We also have that \[|g_n(x) - g_m(x)|^2 \leq (|g_m(x)| + |g_n(x)|)^2 \leq (2h(x))^2.\]

Since \[\lim_{n \to \infty} |g_n(x) - g_m(x)| = 0 \quad \text{for } \mu\text{-a.e. } x, \quad \text{the Dominated Convergence Theorem tells us that}\]

\[\lim_{n \to \infty} \int_{-\pi}^{\pi} |g - g_n|^2 \, d\mu = 0,
\]

which implies that \[\lim_{n \to \infty} \|g - g_n\|_2 = 0.\]

Now, finally, let \(\varepsilon > 0.\)

Choose \(i\) so large that \(\|g_i - g\|_2 < \varepsilon/2\) and \(2^{-i} < \varepsilon/2.\)

Since \(g_i = f_{N_i}\), we have for all \(n \geq N_i:\)

\[\|g - g_n\|_2 \leq \|g - g_i\|_2 + \|g_i - f_n\|_2\]

\[< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\]

So \[\lim_{n \to \infty} \|g - g_n\|_2 = 0, \quad \text{as required.}\]