Chapter 1  Fourier Series

It is extremely useful in analysis to represent functions \( f \) of a real variable as a series. This is especially useful in applied mathematics (e.g. when solving differential equations that arise in physical situations).

Perhaps the first example of this is the Taylor Series of \( f \):

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n
\]

where \( f^{(n)}(0) \) denotes the \( n \)th derivative of \( f \) evaluated at \( 0 \).

However, such a representation of \( f(x) \) by its Taylor Series presupposes that \( f \) can be differentiated any number of times, (and even then the series might not converge).

So we require a series representation for functions \( f \) under much weaker assumptions. In the notes you will find an account of Fourier's study of heat flow in a rectangular plate. A crucial step in his argument requires a manageable series representation for the discontinuous function \( f : [-1, 1] \to \mathbb{R} \) given by

\[
f(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{if } x = 1 \text{ or } x = -1. \end{cases}
\]

He found the remarkable formula (by non-rigorous means)
\[ f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos \left( \frac{(2n-1)\pi x}{2} \right). \]

Since \( \cos \left( \frac{(2n-1)\pi}{2} \right) = \cos \left( \frac{(2n-1)\pi}{2} \right) \neq 0 \) for all \( n \),

this formula is obviously correct for \( x = \pm 1 \).

But what does it mean for arbitrary \( x \)? This is answered in first year sequences and series: for each given \( x \in [-1, 1] \), the series \( \frac{4}{\pi} \sum_{n=1}^{\infty} A_n \) converges, where

\[ A_n = \frac{(-1)^{n-1}}{2n-1} \cos \left( \frac{(2n-1)\pi x}{2} \right), \]

and (as Dirichlet later proved) \(-1 < x < 1\) the sum of the series is 1.

Actually, if one replaces \( x \) by \( x+2 \), then since

\[ \cos \left( \frac{(2n-1)\pi (x+2)}{2} \right) = \cos \left( \frac{(2n-1)\pi x + (2n-1)\pi}{2} \right) = -\cos \left( \frac{(2n-1)\pi x}{2} \right), \]

we see that the sum of the series is -1 for \( 1 < x < 3 \)

and 0 for \( x = 3 \).

Continuing, we see that for all \( x \in \mathbb{R} \), the series

\[ \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos \left( \frac{(2n-1)\pi x}{2} \right) \]

converges to the function (which we still call \( f \)):

\[ f(x) = \begin{cases} 
1 & \text{if } 4m-1 < x < 4m+1, \ m \in \mathbb{Z} \\
0 & \text{if } x = 2m+1, \ m \in \mathbb{Z} \\
-1 & \text{if } 4m+1 < x < 4m+3, \ m \in \mathbb{Z} 
\end{cases} \]
Definition

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is called periodic with period $L$ if
$$f(x) = f(x + L)$$ for all $x \in \mathbb{R}$.

Thus the function above is periodic with period 4. The functions $\cos x$, $\sin x$ are periodic with period $2\pi$.

Definition

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a periodic function with period $L (> 0)$. Then the Fourier series of $f$ is, by definition, the series
$$FS(f)(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{2\pi n x}{L} \right) + b_n \sin \left( \frac{2\pi n x}{L} \right) \right)$$

where $a_0 := \frac{1}{L} \int_{0}^{L} f(x) \, dx$ and, for $n \geq 1$,
$$a_n := \frac{2}{L} \int_{0}^{L} f(x) \cos \left( \frac{2\pi n x}{L} \right) \, dx \quad \text{and} \quad b_n := \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{2\pi n x}{L} \right) \, dx.$$

(The $a_n$, $b_n$ are called the Fourier coefficients of $f$.)

Remark:

The integrals here are the usual Riemann integrals (as studied in your second-year analysis course), so there is an underlying assumption that they exist. One of the main aims of this course will be to develop the theory of Fourier series by regarding (in a suitable set of) functions $f: \mathbb{R} \rightarrow \mathbb{R}$ as elements of a vector space in which the functions
$$1, \cos x, \sin x, \cos(2x), \sin(2x), \ldots, \cos(nx), \sin(nx), \ldots$$
forms a "basis." (We are assuming that all functions under consideration here are periodic with period $2\pi$.)

Then $FS(f)(x)$ is the representation of $f(x)$ in terms of this "basis".

This will necessitate the extension of the notion of integral (to include certain non-Riemann integrable functions)
so that there is a nice theory of when infinite summation and integrals may be exchanged - i.e. when is it valid to assert
\[ \sum x_n \left( \int f_n(x) \, dx \right) \overset{?}{=} \int \left( \sum x_n f_n(x) \right) \, dx. \]
This will also require us to consider the very delicate issue of the convergence of the series $F_S(f)(x)$.

Actually, most of the course is dedicated to the new theory of integration - Lebesgue integration - and we shall return to Fourier series only for the last few lectures.

For the moment, though, we just consider the situation for Riemann integrable functions. The foundational result of the subject was proved in 1829 by Dirichlet.

**Theorem (Dirichlet)**

Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with period $L$. Let $T = 0 < T_1 < \cdots < T_m < T_{m+1} = L$ be a finite set of real numbers in the interval $[0, L]$ and assume further that $f$ is continuous and continuously differentiable on each open interval $(T_i, T_{i+1})$ (for $i = 0, \ldots, m$). Assume also that the right and left limits of both $f$ and $f'$ exist at each point $T_i$ ($i = 0, \ldots, m+1$). Then the Fourier series $F_S(f)(x)$ converges to

(a) $f(x)$ for all $x$ at which $f$ is continuous;

(b) $\frac{1}{2} \left( \lim_{x \to x^+} f(x) + \lim_{x \to x^-} f(x) \right)$ if $f$ is not continuous at $x$.

(Such $x$ lie in the set $\{T_i + nL : n \in \mathbb{Z}, i = 0, \ldots, m \}$.)

**Example**

Let us consider Fourier's example:
So \( L = 4 \), and

\[
f(x) = \begin{cases} 
1 & \text{if } 0 \leq x < 1 \text{ or } 3 < x \leq 4 \\
0 & \text{if } x = 1 \text{ or } x = 3 \\
-1 & \text{if } 1 < x < 3 
\end{cases}
\]

extended by periodicity.

The Fourier coefficients are

\[
a_n = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{2\pi nx}{L} \right) \, dx
\]

and for \( n \geq 1 

\[
a_n = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{2\pi nx}{L} \right) \, dx
= \frac{1}{2} \left[ \int_0^1 \cos \left( \frac{2\pi nx}{L} \right) \, dx + \int_1^3 \cos \left( \frac{2\pi nx}{L} \right) \, dx \\
+ \int_3^4 \cos \left( \frac{2\pi nx}{L} \right) \, dx \right]
= \frac{1}{2} \left[ \left[ \frac{\sin \left( \frac{2\pi nx}{L} \right)}{\frac{2\pi n}{L}} \right]_0^1 + \left[ \frac{\sin \left( \frac{2\pi nx}{L} \right)}{\frac{2\pi n}{L}} \right]_1^3 + \left[ \frac{\sin \left( \frac{2\pi nx}{L} \right)}{\frac{2\pi n}{L}} \right]_3^4 \right]
= \frac{1}{2} \left[ \left[ \sin \left( \frac{2\pi n}{2} \right) \right] - \left[ \sin \left( \frac{6\pi n}{2} \right) \right] + \left[ \sin \left( \frac{12\pi n}{2} \right) \right] - \left[ \sin \left( \frac{18\pi n}{2} \right) \right] \right]
= \frac{1}{\pi n} \left[ \sin \left( \frac{2\pi n}{2} \right) - \sin \left( \frac{3\pi n}{2} \right) \right]
= \frac{2}{\pi n} \left[ \sin \left( \frac{2\pi n}{2} \right) - \sin \left( \frac{3\pi n}{2} \right) \right]
= \begin{cases} 
0 & \text{if } n \text{ is even} \\
\frac{4}{\pi n} & \text{if } n \text{ is odd, say } n = 2n' - 1 \text{ (n' \geq 1)}
\end{cases}
\]

Now check that \( b_n = 0 \) for all \( n \geq 1 \). (This actually follows from the fact that \( f(2-x) = f(2+x) \) for all \( x \in [0,2] \).)
So we apply Dirichlet's theorem with $T_0 = 0$, $T_1 = 1$, $T_2 = 3$, $T_3 = 4$.

The Fourier coefficients, slightly rewritten, are

\[ a_{2n} = 0 \quad \text{for all } n \geq 0, \]
\[ a_{2n-1} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \quad \text{for all } n \geq 1, \]
\[ b_n = 0 \quad \text{for all } n \geq 1. \]

So \( FS(f)(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos \left( \frac{\pi(2n-1)x}{2} \right) \), and the theorem tells us that this converges to \( f(x) \) if \( f \) is continuous at \( x \), i.e., if \( x \neq 1, x \neq 3 \), and, if \( x = 1 \), to

\[ \frac{1}{2} \left( \lim_{t \to 1^+} f(t) + \lim_{t \to 1^-} f(t) \right) = \frac{1}{2} (-1 + 1) = 0 = f(1) \]

and similarly for \( x = 3 \). So we do indeed have

\[ f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos \left( \frac{\pi(2n-1)x}{2} \right) \quad \text{for all } x, \text{ thereby justifying Fourier's formula (22 years after it was discovered).} \]

(Actually, we have only dealt with those \( x \) with \( 0 \leq x \leq 4 \), but the result extends to all \( x \) by periodicity.)

Remark.

The condition that the derivative \( f' \) should exist and be piecewise continuous is annoying, and Dirichlet thought that it could be avoided. However, in 1873, one example of a \( 2\pi \)-periodic, continuous function \( f \colon \mathbb{R} \to \mathbb{R} \) was constructed such that (for \( x = 0 \))

\[ \left( \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos(nx) + b_n \sin(nx) \right) \right) \to \infty \text{ as } N \to \infty, \]

where the \( a_n \)'s and \( b_n \)'s are the Fourier coefficients of \( f \). So \( FS(f)(0) \) does not converge, let alone to \( f(0) \).
But we can still ask whether the Fourier coefficients \( a_n, b_n, \ldots \) of a continuous function determine the function uniquely.

**Pointwise convergence and uniform convergence**

Let \( a, b \in \mathbb{R} \) with \( a < b \), and suppose that we are given a sequence \((f_n)_{n \geq 0}\) where each \( f_n \) is a function \( f_n : [a, b] \to \mathbb{R} \).

In the above we have been dealing with the notion of **pointwise convergence** (where \( f_n(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \)).

In general, we say that the sequence \((f_n)_{n \geq 0}\) converges **pointwise** to a function \( f : [a, b] \to \mathbb{R} \), if for each \( x \in [a, b] \), \( f_n(x) \to f(x) \) as \( n \to \infty \). This means that for each \( x \in [a, b] \), and each \( \varepsilon > 0 \), there is an \( N \) (which depends on both \( \varepsilon \) and \( x \)) such that \( |f_n(x) - f(x)| < \varepsilon \) for all \( n \geq N \).

We say that \((f_n)_{n \geq 0}\) converges **uniformly** to \( f \), if the \( N \) above can be taken to depend only on \( \varepsilon \), and not on \( x \).

In other words, \( f_n \to f \) uniformly, if for all \( \varepsilon > 0 \), there exists \( N \) such that \( |f_n(x) - f(x)| < \varepsilon \) for all \( n \geq N \) and for all \( x \in [a, b] \).

Equivalently,

\[
\sup_{x \in [a, b]} |f_n(x) - f(x)| \to 0 \quad \text{as} \quad n \to \infty.
\]

(It's trivial that uniform convergence implies pointwise convergence, but the converse is false — see problem 3 of Example sheet 1.)

Now Dirichlet's theorem concerns the pointwise convergence of the partial sums of a Fourier series and it turns out that this cannot be improved to uniform convergence.
However, if one averages terms, things work out much better.

**Definition**

Let \( (a_n)_{n \geq 1} \) be any sequence of real numbers. The sequence \((\sigma_n)_{n \geq 0}\) defined by

\[
\sigma_n := \frac{a_1 + a_2 + \ldots + a_n}{n}
\]

is called the Cesàro average of \((a_n)_{n \geq 0}\), and if \( \lim_{n \to \infty} \sigma_n \) exists, then \( \lim_{n \to \infty} a_n \) is called the Cesàro limit of \((a_n)_{n \geq 0}\).

**Exercise**

If \( \lim_{n \to \infty} \sigma_n \) exists, then so does the Cesàro limit and has the same value.

**Example**

The sequence \(0, 2, 0, 2, \ldots = (1 - (-1)^n)_{n \geq 0}\) has no limit, but has Cesàro limit equal to 1.

One can extend this notion to sequences of functions in general, but we just do it for the partial sums of Fourier series.

So let \( f : \mathbb{R} \to \mathbb{R} \) be a periodic function with period 2\(\pi\) and let \( a_n + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \) be its Fourier series.

Define

\[
S_m(f, x) := \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx)) \quad (n \geq 0)
\]

as \((the \ m^{th} \ partial \ sum \ of \ \sum_{n=0}^{\infty} f(x))\). Now let

\[
\sigma_n(f, x) = \frac{1}{n} \left( S_0(f, x) + S_1(f, x) + \cdots + S_{n-1}(f, x) \right) \quad (n \geq 1)
\]

be the Cesàro average.
Fejér’s Theorem

Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is continuous and periodic with period \( 2\pi \). Then the sequence of functions \((\varphi_n(f,x))_{n \geq 1}\) converges uniformly to \( f \), i.e.,

\[
\sup_{x \in \mathbb{R}} |\varphi_n(f,x) - f(x)| \to 0 \quad \text{as} \quad n \to \infty.
\]

(The continuity of \( f \) is necessary here because each function \( \varphi_n(f,x) \) is obviously continuous and one can show that the uniform limit of a sequence of continuous functions is itself a continuous function.)

The proof is in the printed notes and is non-examinable.

It uses the complete notation for Fourier series (via the identities \( \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \), \( \sin \theta = -i \left( \frac{e^{i\theta} - e^{-i\theta}}{2} \right) \)) which can also be found in the notes.

Corollary

Suppose that \( f, g : \mathbb{R} \to \mathbb{R} \) are continuous, \( 2\pi \)-periodic functions with the same Fourier coefficients. Then \( f = g \) (i.e., \( f(x) = g(x) \) for all \( x \in \mathbb{R} \)).

Proof.

Suppose \( f \neq g \). Choose \( x_0 \in \mathbb{R} \) such that \( f(x_0) \neq g(x_0) \). Let

\[ \varepsilon = \frac{1}{2} |f(x_0) - g(x_0)|, \]

so that \( \varepsilon > 0 \). Choose \( N \) so that for all \( n \geq N \),

\[ \sup_{x \in \mathbb{R}} |\varphi_n(f,x) - f(x)| < \varepsilon \quad \text{and} \quad \sup_{x \in \mathbb{R}} |\varphi_n(g,x) - g(x)| < \varepsilon, \]

(using Fejér’s Theorem). Now by hypothesis, \( \varphi_n(f,x) = \varphi_n(g,x) \) for all \( n \) and all \( x \in \mathbb{R} \). Let \( A = \varphi_N(f,x_0) = \varphi_N(g,x_0) \).

Then

\[ |A - f(x_0)| < \varepsilon \quad \text{and} \quad |A - g(x_0)| < \varepsilon, \]

so

\[ |f(x_0) - g(x_0)| = |f(x_0) - A + A - g(x_0)| \leq |f(x_0) - A| + |A - g(x_0)| < 2\varepsilon. \]

A contradiction.
Remarks on Example sheet 1

There's not much left to do for 1(c).

I have used the range of integration $\int_{0}^{L} \cdots \, dt$ for the Fourier coefficients, but by periodicity it doesn't matter which range we take as long as it has length $L$.

E.g. in question 2, where $L = 2\pi$, it's better to take $\int_{-\pi}^{\pi} \cdots \, dt$ rather than $\int_{0}^{2\pi} \cdots \, dt$. 