On Exchange Properties for Coxeter Matroids and Oriented Matroids

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Abstract

We introduce new basis exchange axioms for matroids and oriented matroids. These new axioms are special cases of exchange properties for a more general class of combinatorial structures, Coxeter matroids. We refer to them as "properties" in the more general setting because they are not all equivalent, as they are for ordinary matroids, since the Symmetric Exchange Property is strictly stronger than the others. The weaker ones constitute the definition of Coxeter matroids, and we also prove their equivalence to the matroid polytope property of Gelfand and Serganova.

The terminology in the present paper follows [**BG**, **BR**] (though we prefer to use the name 'Coxeter matroids' rather than 'WP-matroids,' as used in these papers); see also the forthcoming book [**BGW1**].

The cited publications also contain all the necessary background material. For more detail, refer to books [**We**],[**Wh**], [**O**] and [**R**] for the systematic exposition of matroid theory and theory of Coxeter complexes.

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1 Exchange properties for matroids

Matroids. The following is well-known (see for example [**O**]):

Theorem 1.1 *Let $\mathcal{B}$ be a non-empty collection of subsets of E. Then the following are equivalent:*

(1) *For every $A, B \in \mathcal{B}$ and $a \in A \setminus B$ there exists $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\} \in \mathcal{B}$ (the* **Exchange Property***).*

(2) *For every $A, B \in \mathcal{B}$ and $a \in A \setminus B$ there exists $b \in B \setminus A$ such that $B \setminus \{b\} \cup \{a\} \in \mathcal{B}$ (the* **Dual Exchange Property***).*

(3) *For every $A, B \in \mathcal{B}$ and $a \in A \setminus B$, there exists $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\} \in \mathcal{B}$ and $B \setminus \{b\} \cup \{a\} \in \mathcal{B}$ (the* **Symmetric Exchange Property***).*

A pair $M = (\mathcal{B}, E)$ is called a *matroid on E* if $\mathcal{B}$ satisfies one of the conditions (**1**), (**2**), (**3**). The elements of $\mathcal{B}$ are called the *bases* of the matroid M. It is easy to prove that the bases of a matroid are incomparable, and moreover are of the same cardinality which is called the *rank* of M. We shall say that the basis $B = A \setminus \{a\} \cup \{b\}$ in the Exchange Property is obtained from the basis A by the transposition $t = (a, b)$, and write $B = tA$. We also say that the bases A and B are *adjacent*. It will be convenient for us to identify E with the set $[n] = \{1, 2, \ldots, n\}$.

One of the purposes of the present paper is to show that the Exchange Property is equivalent to some other, apparently weaker, versions of the exchange condition for bases. These exchange properties naturally arise in the more general setting of Coxeter matroids.

The Exchange Property **(1)** turns out to be equivalent to what we call the **Fully Symmetric Exchange Property**:

(4) *For every $A, B \in \mathcal{B}$, $A \neq B$, there exist $a \in A \setminus B$ and $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\} \in \mathcal{B}$ and $B \setminus \{b\} \cup \{a\} \in \mathcal{B}$.*

At first, we believed this property to be a new form of the basis exchange axiom for ordinary matroids. We recently learned that A. Kelmans proved the equivalence of this property and another, seemingly weaker one to the usual matroid axioms in 1973. He introduced the following property

(5) *For every* $A, B \in \mathcal{B}$, $A \neq B$, *there exist* $a \in A \setminus B$, $b \in B \setminus A$ *and* $b' \in B \setminus A$ *such that* $A \setminus \{a\} \cup \{b\} \in \mathcal{B}$ *and* $B \setminus \{b'\} \cup \{a\} \in \mathcal{B}$.

and proved the following

Theorem 1.2 (Kelmans [K]) *Let* $\mathcal{B}$ *be a non-empty collection of subsets of* E. *Then conditions* (**1**), (**4**), *and* (**5**) *are equivalent.*

Since these result is not widely known, at least in the West, we present Kelmans' proof here.

Proof. By Theorem 1.1, conditions (**1**), (**2**), (**3**) are equivalent. Clearly (**3**) implies (**4**) and (**4**) implies (**5**). Therefore we need only show that (**5**) implies (**2**). We assume that (**5**) holds, and prove (**2**) by induction on $|A \setminus B|$. First we note that it is easy to see that (**5**) implies that all members of $\mathcal{B}$ have the same cardinality. Now, if $|A \setminus B| = 1 = |B \setminus A|$, the statement (**2**) is clearly true. Let $|A \setminus B| = n \geq 2$. By (**5**), there are $a \in A \setminus B$, $b \in B \setminus A$ and $b' \in B \setminus A$ such that $A' := A \setminus \{a\} \cup \{b\} \in \mathcal{B}$ and $B \setminus \{b'\} \cup \{a\} \in \mathcal{B}$. Now, $A' \setminus B = (A \setminus B) \setminus \{a\} \subset A \setminus B$ and $B \setminus A' = (B \setminus A) \setminus \{b\} \subset B \setminus A$. Since $|A' \setminus B| = n-1$, by the induction hypothesis, condition (**2**) holds for A' and B, so for every $c \in (A \setminus B) \setminus \{a\}$, there exists $d \in (B \setminus A) \setminus \{b\}$ such that $B \setminus \{d\} \cup \{c\} \in \mathcal{B}$. Since $B \setminus \{b'\} \cup \{a\} \in \mathcal{B}$ also, we have condition (**2**) for A and B. □

Maximality Property. A definition of matroid in terms of the greedy algorithm was first given by Boruvka (1926), and rediscovered many times, including [**G**] and [**GS2**]. This is restated in the theorem below.

Let $\mathcal{P}_k = \mathcal{P}_k(n)$ be the set of all k-element subsets in $[n]$. We introduce a partial ordering $\leqslant$ on $\mathcal{P}_k$ as follows. Let $A, B \in \mathcal{P}_k$, where

$$A = \{i_1, \ldots, i_k\},\ i_1 < i_2 < \cdots < i_k$$

and

$$B = \{j_1, \ldots, j_k\},\ j_1 < j_2 < \cdots < j_k,$$

then we set

$$A \leqslant B \iff i_1 \leqslant j_1, \ldots, i_k \leqslant j_k.$$

Let $W = Sym_n$ be the group of all permutations of $[n]$. Then we can associate an ordering of $\mathcal{P}_k$ with each $w \in W$ by denoting $wA = \{wa : a \in A\}$ and putting

$$A \leqslant^w B \text{ if and only if } w^{-1}A \leqslant w^{-1}B.$$

Clearly $\leqslant^1$ is just $\leqslant$.

If in this definition we set $k = 1$, it will be convenient for us to write $i \leqslant^w j$ instead of $\{i\} \leqslant^w \{j\}$. This simply means that $w^{-1}(i) \leq w^{-1}(j)$, in othe words, i precedes j in the bottom row of the standard two-rowed notation for the permutation w:

$$w = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}.$$

Thus, the permutation w can be interpreted as the reordering

$$i_1 <^w i_2 <^w \cdots <^w i_n$$

of the set $[n]$.

Theorem 1.3 *Let $\mathcal{B} \subseteq \mathcal{P}_k$ be a set of k-subsets. Then $\mathcal{B}$ is the collection of bases of a matroid if and only if $\mathcal{B}$ satisfies the following* **Maximality Property:**

> *for every $w \in Sym_n$ the collection $\mathcal{B}$ contains a unique maximal member A in $\mathcal{B}$ with respect to $\leqslant^w$: $B \leqslant^w A$ for all $B \in \mathcal{B}$.*

We call A the *w-maximal* element in $\mathcal{B}$.

Increasing Exchange Property. The Exchange Property for matroids is also equivalent to the following property which we call the **Increasing Exchange Property**) of bases of a matroid.

(6) *Let $\mathcal{B}$ be a collection of subsets in $[n]$ of the same cardinality, A_1, A_2 two different sets from $\mathcal{B}$ and $w \in Sym_n$ an arbitrary permutation. Then there is a transposition (a_1, a_2) with $a_1 < a_2$ in w and such that one of A_1, A_2, say A_i, contains a_1, does not contain a_2 and $(a_1, a_2)A_i = A_i \setminus \{a_1\} \cup \{a_2\}$ also belongs to $\mathcal{B}$.*

In plain language this property means that for every two different bases in $\mathcal{B}$ and a permutation w of $[n]$, one of these two bases can be increased with respect to w by a transposition.

Theorem 1.4 *For a set $\mathcal{B}$ of k-element subsets of the set $[n]$, the Increasing Exchange Property is equivalent to the Maximality Property, as well as the other exchange properties of a matroid..*

Proof. We already know that the Fully Symmetric Exchange Property and the Maximality Property are equivalent. So we need to prove only that the Fully Symmetric Exchange Property implies the Increasing Exchange Property and that the Increasing Exchange Property implies the Maximality Property.

Assume that $\mathcal{B}$ satisfies the Fully Symmetric Exchange Property. Then, given two sets A_1 and A_2 from $\mathcal{B}$, we have elements $a_1 \in A_1$ and $a_2 \in A_2$ such that for the transposition $t = (a_1, a_2)$ both sets tA_1 and tA_2 belong to $\mathcal{B}$. Let now $w \in Sym_n$ be an arbitrary reordering and assume $a_i <^w a_j$, $i, j = 1, 2$. Then $A_i <^w tA_i$ and tA_i belongs to $\mathcal{B}$, thus proving the Increasing Exchange Property.

Assume now that $\mathcal{B}$ satisfies the Increasing Exchange Property. We want to prove the Maximality Property for $\mathcal{B}$. Assuming the contrary, let the Maximality Property fail, and, for $w \in Sym_n$, let A_1 and A_2 be two different w-maximal elements of $\mathcal{B}$. Then if A_i and the transposition $t = (a_1, a_2)$ are as in the Increasing Exchange Property, $tA_i = A_i \smallsetminus \{a_1\} \cup \{a_2\}$ is obviously w-bigger than A_i, which contradicts the maximal choice of A_i. □

The Increasing Exchange Property yields the following property of the ordering on the collection of bases which is interesting from the point of view of the general theory of Coxeter matroids (cf. Theorem 4.3 below).

Theorem 1.5 *Let $\mathcal{B}$ be the collection of bases of a matroid on $[n]$. Let w be an arbitrary permutation in $W = Sym_n$ and B the w-maximal element of $\mathcal{B}$. If $A \in \mathcal{B}$ is an arbitrary base, then there exists a sequence of transpositions*

$$t_1 = (c_1, d_1),\ t_2 = (c_2, d_2),\ \ldots,\ t_s = (c_s, d_s)$$

such that $c_i <^w d_i$ for $i = 1, 2, \ldots, s$, all the sets $A_i = t_i \cdots t_1 A$ belong to $\mathcal{B}$, and $B = t_s \cdots t_1 A = A_s$.

In plain language it means that if B is the w-maximal basis, then any other basis can be connected with B by a w-increasing chain of adjacent bases.

Proof. In the Increasing Exchange Property applied to the bases A and B and the ordering w, the w-maximal basis B cannot be further increased, so there is a transposition t_1 such that $A <^w t_1 A$. Repeating the same procedure inductively, we eventually get a desired w-increasing sequence of adjacent bases connecting A and B. □

2 Coxeter Matroids

In this section we use the terminology of the theory of Coxeter groups and Coxeter complexes. See [**R**] for the systematic development of the theory and [**BG**, **BR**] for its use in the theory of matroids.

The Maximality Property and the Symmetric Exchange Property. Let W be a Coxeter group, P a finite parabolic subgroup in W, $W^P = W/P$, and $\mathcal{M}$ a subset of W^P. The symbol $\leqslant$ denotes the Bruhat ordering on W and the induced Bruhat ordering on W^P. For two cosets $A, B \in W^P$ and element $w \in W$, $A \leqslant^w B$ stands for $w^{-1}A \leqslant w^{-1}B$. We say that $\mathcal{M} \subseteq W^P$ satisfies the **Maximality Property** if,

> *for any $w \in W$, there is a unique $A \in \mathcal{M}$ such that, for all $B \in \mathcal{M}$, $B \leqslant^w A$.*

The Maximality Property is one of the (equivalent) definitions of a Coxeter matroid: a set $\mathcal{M} \leqslant W^P$ is a **Coxeter matroid** for W and P if and only if it satisfies the Maximality Property.

It is convenient for our purposes to identify the group W with its Coxeter complex (which we denote by the same letter W) and to treat elements of W as chambers, cosets in W^P as residues, etc.

Ordinary matroids of rank k on n letters constitute a special case of a Coxeter matroid. Each basis B of a matroid $\mathcal{B}$ can be identified with a coset in W^P for $W = Sym_n$ and

$$P = \langle (12), (23), \ldots, (k-1, k), (k+1, k+2), \ldots, (n-1, n) \rangle$$

(notice that P is the stabilizer of the k-set $\{1, 2, \ldots, k\}$). Then $\mathcal{M}$ is the collection of cosets corresponding to $\mathcal{B}$. The obvious and straightforward translation of the Fully Symmetric Exchange Property for ordinary matroids into the more general language of Coxeter matroids is the following **Symmetric Exchange Property for Coxeter matroids**:

> *For any two distinct cosets A and B in $\mathcal{M}$ there is a wall Σ separating them and such that the reflections sA and sB of A and B in Σ belong to $\mathcal{M}$.*

Note that there is no obvious translation of the Symmetric Exchange Property for ordinary matroids into the more general Coxeter matroids, hence we have dropped the adjective "Fully" here.

Theorem 2.1 *Let W be a Coxeter group and P a finite parabolic subgroup in W. If a finite set $\mathcal{M} \subseteq W^P$ satisfies the Symmetric Exchange Property then $\mathcal{M}$ satisfies the Maximality Property.*

Proof: Assume that $\mathcal{M}$ satisfies the Symmetric Exchange Property but the Maximality Property fails in $\mathcal{M}$ for $w \in W$, i.e. there are two cosets $A, B \in \mathcal{M}$ maximal in $\mathcal{M}$ with respect to the ordering $\leqslant^w$. Let Σ be the wall given by the Symmetric Exchange Property for A and B. The chamber w of the Coxeter complex W lies in one of the halfcomplexes R and L bounded by Σ. We can assume without loss of generality that A and w lie in the same halfcomplex R of W. Let $s \in W$ be the reflection in the wall Σ. Then $A' = sA$ lies on the opposite side of the wall Σ, and also, by the Symmetric Exchange Property, $A' \in M$. If now $w_{A'}$ is the $\leqslant^w$-minimal chamber of the coset A', and Γ' is a geodesic gallery from w to $w_{A'}$, the folding of the Coxeter complex W onto the halfcomplex R sends Γ to a gallery Γ from w to some chamber in the coset A. But this means that $A \leqslant^w A'$, contrary to our maximal choice of A. □

Flag-matroids. Unfortunately, it is not true that, for an arbitrary finite Coxeter group W, parabolic subgroup $P < W$ and set $\mathcal{M} \subseteq W^P$, the Symmetric Exchange Property is equivalent to the Maximality Property. For example, it is not true for flag-matroids (see [**GS2**] for the interpretation of flag-matroids as Coxeter matroids).

Indeed, consider the ordinary matroid of rank 3 on 4 points whose bases are 123, 124, 134, its rank 2 'strong map image' (see [**Wh**]) whose bases are 12, 14, 23, 24, 34, and the rank 1 strong map image whose bases are 1, 3, 4. Then a flag of bases such as 1, 12, 123 will be abbreviated as the flag 123. There are then 10 flags, 123, 124, 142, 143, 321, 341, 412, 413, 421, 431. This flag matroid fails to satisfy the Symmetric Exchange Property for the basis pair 123 and 413. Each of the transpositions (12), (14), (23), (24), (34) sends at least one of these two bases to a non-basis (where 'basis' now means 'flag of bases'). Furthermore, both are on the same side of the wall corresponding to the transposition (13). These are all six of the reflections in the Coxeter group $A_3 = Sym_4$, so the Symmetric Exchange Property has failed.

The Increasing Exchange Property for Coxeter Matroids. The very straightforward translation of the Increasing Exchange Property for ordinary matroids in the general setting of Coxeter matroids reads as follows. Let W be a Coxeter group and P a finite parabolic subgroup in W. We shall say that a set $\mathcal{M} \subseteq W^P$ satisfies the **Increasing Exchange Property** if

> *for any two distinct cosets A_1 and A_2 in $\mathcal{M}$ and an arbitrary element $w \in W$ there is a wall Σ, such that one of the cosets*

A_1 and A_2, say, A_i, lies on the same side of Σ as the chamber w and the reflection sA_i of A_i in the wall Σ belongs to $\mathcal{M}$.

The Increasing Exchange Property can be easily restated in terms of the Bruhat ordering.

The set $\mathcal{M} \subseteq W^P$ satisfies the Increasing Exchange Property if and only if for any two distinct cosets A_1 and A_2 in $\mathcal{M}$ and element $w \in W$ there is a reflection $s \in W$ such that for one of the cosets A_1 and A_2, say, A_i, the coset sA_i belongs to $\mathcal{M}$ and $A_i \leqslant^w sA_i$.

We shall prove (Theorem 4.1) that the Increasing Exchange Property is equivalent to the Maximality Property.

3 Root systems

We shall use the technique developed in [**GS2**, § 8.3] in the proof of the main result of the present paper, Theorem 4.1.

Notation. Concerning reflection groups and root systems we have adapted the terminology from [**H**].

Let V be the space of the reflection representation for the Coxeter group W and $(\ ,\ )$ a W-invariant scalar product in V. We denote by Φ the root system and by $\Pi = \{\rho_1, \dots, \rho_n\}$ the simple root system corresponding to the system of standard generators $r_1, \dots, r_n$ of W. We denote by Φ^+ the system of positive roots corresponding to Π. We say that a root ρ is *w-positive* for $w \in W$ if $\rho \in w\Phi^+$, and *w-simple* if $\rho \in w\Pi$.

Orbits in the space of the reflection representation. Now let $J \neq \emptyset$ be a subset of $I = \{1, 2, \dots, n\}$ and $P = \langle r_i \mid i \in I \smallsetminus J\rangle$ the corresponding parabolic subgroup in W. Consider the point $\omega_J \in V$ defined by

$$\frac{(\omega_J, \rho_i)}{(\rho_i, \rho_i)} = \begin{cases} -1 & \text{for} \quad \rho_i \in J, \\ \ \ 0 & \text{for} \quad \rho_i \notin J. \end{cases}$$

Since P is the subgroup fixing ω_J, we can define a mapping $\delta : W^P \longrightarrow V$ that sends wP to $w\omega_J$. We denote $\delta(A)$ by δ_A for all $A \in W^P$. The mapping δ identifies the factor set W^P with the orbit $W\omega_J$.

Lemma 3.1 *Let $\rho \in \Phi$ be a root and $s \in W$ the corresponding reflection. Then, for any two points δ_A and δ_B in $W\omega_J$, the following conditions are equivalent:*

(1) $B = sA$.

(2) $\delta_B = s\delta_A$.

(3) $\delta_B - \delta_A = c\rho$ *for some scalar* c.

Proof. Equivalence of (1) and (2) is obvious. Statement (3) follows from (1) immediately, by the well-known formula for reflection in the hyperplane normal to ρ:

$$s\delta_A = \delta_A - 2\frac{(\delta_A, \rho)}{(\rho, \rho)}\rho.$$

Now assume (3). Since the vectors δ_A and δ_B have equal lengths, the hyperplane Σ normal to the edge $[\delta_A, \delta_B]$ and cutting it at the midpoint passes through the origin. Obviously Σ is the mirror of the reflection s corresponding to ρ and $s\delta_A = \delta_B$, $sA = B$. □

Fundamental domain and Coxeter complex. The open convex polyhedral cone

$$D = \{\, \chi \in V \mid (\chi, \rho) > 0 \text{ for all } \rho \in \Pi \,\}$$

is called the *standard fundamental domain* for W. Its closure

$$\overline{D} = \{\, \chi \in V \mid (\chi, \rho) \geqslant 0 \text{ for all } \rho \in \Pi \,\}$$

is a closed convex polyhedral cone whose facets (i.e. faces of maximal dimension) lie on the hyperplanes $(\chi, \rho_i) = 0$ for all simple roots $\rho_1, \dots, \rho_n$ in Π. Therefore we can label these facets by the reflections $r_1, \dots, r_n$ corresponding to the simple roots. It is well-known from the theory of Coxeter groups [**H**] that $D \cap wD = \emptyset$ for all $w \in W$, $w \neq 1$, that W acts simply transitively on the set $\{\, wD \mid w \in W \,\}$, and that V is the union of the closed polyhedral cones $w\overline{D}$ for $w \in W$. Open polyhedral cones wD, $w \in W$, are called *chambers.* We can transfer, via the action of the element w, labelling of facets from D to wD, and say that two chambers A and B are r_i-adjacent if their closures have a common facet labelled r_i. Denote

$$E = \{\, \chi \in V \mid (\chi, \rho) < 0 \text{ for all } \rho \in \Pi \,\}.$$

It can be shown that $E = -D$ is also a chamber. Notice that, for any non-empty subset $J \subseteq I$, the point ω_J belongs to the closure $\overline{E}$ of E. After that the set of all chambers attains a structure of a chamber complex which is canonically and W-equivariantly isomorphic to the Coxeter complex for W. The group W itself also has the natural structure of a chamber complex: elements of W are chambers and two chambers u and v are r_i-adjacent if and only if $u = r_i v$. This chamber complex is isomorphic to the Coxeter complex for W. We shall identify the three complexes and freely use the combinatorial, geometric and group-theoretical languages. Moreover, it will be most convenient for us to denote all three complexes by the letter W and identify the chamber E with the identity element $1 \in W$.

Notice that, under the above conventions, if $s \in W$ is a reflection then two chambers C_1 and C_2 in V are separated by the mirror of the reflection s (i.e. by the hyperplane of s-invariant points in V) if and only if the corresponding chambers c_1 and c_2 of the Coxeter complex for W are separated by the wall of the same reflection s.

Orderings of $W\omega_J$. For every $w \in W$ we shall define an ordering $\prec^w$ of V as follows. Let C_w be the convex cone in V consisting of vectors $y = \sum_{i=1}^n c_i w \rho_i$ such that $c_i \geqslant 0$ for $i = 1, 2, \dots, n$. Thus C_w is the convex polyhedral cone spanned by the system of w-positive roots. We define the ordering $\prec^w$ on V by putting $x \prec^w y$ if $y - x \in C_w$. Notice that, obviously, $x \prec^w y$ if and only if $w^{-1}x \prec w^{-1}y$.

The restriction of this ordering to $W\omega_J$ agrees with the ordering $\leqslant^w$ on W^P in the following sense.

Lemma 3.2 *If $A, B \in W^P$ and $A \leqslant^w B$ then $\delta_A \prec^w \delta_B$.*

Proof. Since $A \leqslant^w B$ means $w^{-1}A \leqslant w^{-1}B$ and $\delta_A \prec^w \delta_B$ holds if and only if $\delta_{w^{-1}A} \prec \delta_{w^{-1}B}$, it will suffice to prove the lemma in the case $w = 1$. If $A \leqslant B$ then, by the definition of the Bruhat ordering, there is a sequence $s_1, \dots, s_t$ of reflections such that

$$A \leqslant s_1 A \leqslant s_2 s_1 A \leqslant \cdots \leqslant s_t \cdots s_1 A$$

and

$$B = s_t \cdots s_1 A.$$

It will be enough to prove that if $s \in W$ is the reflection corresponding to a positive root ρ then $A \leqslant sA$ implies $\delta_{sA} - \delta_A = c\rho$ for some non-negative scalar c. This is a consequence of the following more general lemma.

Lemma 3.3 *Assume that $A, B \in W^P$ and $\delta_B - \delta_A = c\rho$ for a positive root ρ. Then $A \leqslant B$ if and only if $c \geqslant 0$.*

Proof. The equality case is obvious: $A = B$ if and only if $\delta_A = \delta_B$. So assume that A and B are distinct. Let s be the reflection corresponding to the root ρ, then we know from Lemma 3.1 that $B = sA$.

Notice that $A \leqslant sA$ means that, in the Coxeter complex for W, the residue A and the chamber $\overline{E}$ lie on one side of the wall Σ of reflection s, and sA on the other. But then, by the discussion above, the point δ_A lies on the same side of the mirror Σ of reflection s as the point $\delta_P = \omega_J$. But we have chosen ω_J to lie in the closure of the chamber $E = -D$ opposite to the standard fundamental domain D, therefore the reflection s moves the point δ_A from the halfspace bounded by Σ and containing the chamber $-D$ to the half space containing the standard fundamental domain D. Therefore $\delta_{sA} - \delta_A = c\rho$ for a positive scalar coefficient c. This argument can be easily reversed, completing the proofs of Lemmas 3.3 and 3.2. □ □

We can restate Lemma 3.3 in the following form, which will be used in the proof of Theorem 4.1.

Lemma 3.4 *Assume that* $A, B \in W^P$ *and* $\delta_B - \delta_A = c\rho$ *for a root* $\rho \in \Phi$. *Then* $A \leqslant^w B$ *if and only if* $0 \prec^w \delta_B - \delta_A$.

The converse of Lemma 3.2 is not in general true. Indeed, it does not follow from $\delta_A \prec^w \delta_B$ that $A \leqslant^w B$; an easy counterexample is given in [**BGW3**]; see also [**D**] for a discussion of geometric interpretations of the Bruhat ordering. Unfortunately, the original proof of one of the main results in the theory of Coxeter matroids, Theorem 8.1 in the paper by I. M. Gelfand and V. V. Serganova [**GS2**], relies on this converse and for this reason has to be amended. We have incorporated the corrected proof of the Gelfand-Serganova Theorem in our Theorem 4.1 below.

4 Matroid polytopes

We say that a polytope Δ in V is a *matroid polytope* if Δ is convex and its edges are parallel to the roots in Φ.

With any subset $\mathcal{M} \subseteq W^P$ we associate a polytope $\Delta_{\mathcal{M}}$, the convex hull of points in $\delta(\mathcal{M})$. Notice that, since the group W acts transitively on the set $W\omega_J$, all points in $W\omega_J$ are vertices of the convex hull of $W\omega_J$. Therefore the set $\delta(\mathcal{M})$ is exactly the set of vertices of $\Delta_{\mathcal{M}}$.

Theorem 4.1 *Let* W *be a finite Coxeter group,* P *a parabolic subgroup in* W, $\mathcal{M}$ *a subset in* W^P, *and* $\Delta = \Delta_{\mathcal{M}}$ *the polytope associated with* $\mathcal{M}$.

Then the following conditions are equivalent.

(1) $\mathcal{M}$ *is a Coxeter matroid.*

(2) Δ *is a matroid polytope.*

(3) $\mathcal{M}$ *satisfies the Increasing Exchange Property.*

Proof. (1) *implies* (2). Assume first that $\mathcal{M}$ is a Coxeter matroid. Let l be an edge with vertices δ_A and δ_B that is not parallel to any root. Consider a linear function $f : V \longrightarrow \mathbb{R}$ which is constant on l and takes smaller values on the other points of Δ. Since l is not parallel to any root, we can also can assume without loss of generality that f is not vanishing on any root in Φ. One can easily find a total ordering $\leq$ of V such that, for all $\alpha, \beta \in V$, $\alpha \leq \beta$ implies $f(\alpha) \leqslant f(\beta)$. It is well-known [**H**, Theorem 1.3] that there is a unique simple system of roots $\tilde{\rho}_1, \tilde{\rho}_2, \ldots, \tilde{\rho}_n$ such that $0 \leq \tilde{\rho}_i$, $i = 1, 2, \ldots, n$. Since $f(\tilde{\rho}_i) \neq 0$, we have $f(\tilde{\rho}_i) > 0$, $i = 1, 2, \ldots, n$.

Another basic fact from the theory of Coxeter groups, [**H**, Theorem 1.4], asserts that the group W acts transitively on the set of all simple root systems. Therefore there is an element $w \in W$ which sends $\{\rho_1, \rho_2, \ldots, \rho_n\}$ to $\{\tilde{\rho}_1, \tilde{\rho}_2, \ldots, \tilde{\rho}_n\}$. Then for any coset $C \in \mathcal{M}$ distinct from A we have $f(\delta_C) \leqslant f(\delta_A)$ and the vector $\delta_C - \delta_A$ has at least one negative coefficient with respect to $\{\tilde{\rho}_1, \tilde{\rho}_2, \ldots, \tilde{\rho}_n\}$. But this makes impossible the inequality $A \leqslant^w C$, because the latter implies, by Lemma 3.2, that $\delta_C - \delta_A$ is a non-negative linear combination of the roots $\tilde{\rho}_i$. Therefore, by the Maximality Property, A is the w-maximal element of $\mathcal{M}$. But the same arguments can be applied to the vertex δ_B, and yield that B is also the w-maximal element of $\mathcal{M}$, a contradiction to the Maximality Property.

(2) *implies* (3). Assume now that Δ is a matroid polytope. Let A_1 and A_2 be two arbitrary distinct cosets in $\mathcal{M}$ and $w \in W$. Let $\delta_{B_1}, \ldots, \delta_{B_l}$ be the vertices of Δ adjacent to δ_{A_1}, and $\delta_{C_1}, \ldots, \delta_{C_m}$ the vertices adjacent to δ_{A_2}. Denote $\beta_i = \delta_{B_i} - \delta_{A_1}$, $i = 1, 2, \ldots, l$, $\gamma_j = \delta_{C_j} - \delta_{A_2}$, $j = 1, 2, \ldots, m$. In view of Lemma 3.4 it will suffice to prove that one of the edges β_i, γ_j is w-positive.

Assume the contrary, let all $\beta_i \prec^w 0$ and $\gamma_j \prec^w 0$ for all i and j. The convex polytope Δ is contained in the convex polyhedral cone spanned by the edges emanating from δ_{A_1}. In turn, these edges are contained in the convex cone $\Gamma = \{\, \chi \in V \mid \chi \prec^w \delta_{A_1} \,\}$. Therefore $\Delta \subseteq \Gamma$ and $\delta_A \prec^w \delta_{A_1}$ for all vertices δ_A of Δ. In particular, $\delta_{A_2} \prec^w \delta_{A_1}$. But exactly the same argument can be applied to δ_{A_2} yielding $\delta_{A_1} \prec^w \delta_{A_2}$. Therefore $\delta_{A_1} = \delta_{A_2}$ and $A_1 = A_2$, a contradiction.

(3) *implies* (1). Assume now that $\mathcal{M}$ satisfies the Increasing Exchange Property. We want to prove the Maximality Property for $\mathcal{M}$. Assume to the contrary that the Maximality Property fails, and, for $w \in W$, A_1 and A_2 are two different w-maximal elements of $\mathcal{M}$. Then if A_i and reflection s are as in the Increasing Exchange Property, sA_i is w-bigger than A_i, which contradicts the maximal choice of A_i. □

It immediately follows from Lemma 3.1 that Theorem 4.1 can be restated in the following form, which has already appeared implicitly in the proof of Theorem 4.1.

Theorem 4.2 *A subset $\mathcal{M} \subseteq W^P$ is a Coxeter matroid if and only if for any pair of adjacent vertices δ_A and δ_B of $\Delta_{\mathcal{M}}$ there is a reflection $s \in W$ such that $s\delta_A = \delta_B$ (and also $s\delta_B = \delta_A$, $sB = A$ and $sA = B$).*

The following property of Coxeter matroids is an obvious corollary of the Increasing Exchange Property (Theorem 4.1), and its proof is exactly the same as the proof of its special case, Theorem 1.5.

Theorem 4.3 *Let $\mathcal{M}$ be a Coxeter matroid in W^P for a finite Coxeter group W and a parabolic subgroup P. Let $w \in W$ and let B be the unique w-maximal coset in $\mathcal{M}$. If $A \in \mathcal{M}$ is an arbitrary coset distinct from B then there exists a sequence of reflections $s_1, s_2, \ldots, s_m$ such that all the cosets $A_i = s_i \cdots s_2 s_1 A$, $i = 1, 2, \ldots, m$, belong to $\mathcal{M}$, $A_m = B$ and*

$$A <^w A_1 <^w A_2 <^w \cdots <^w A_m = B.$$

Fans of convex cones. It is interesting to compare Theorem 4.2 with the following result from [**BR**] which is stated in terms of the combinatorial geometry of the Coxeter complex for the Coxeter group W.

Recall [**R**] that a subset X of the Coxeter complex W is called *convex* if any geodesic gallery connecting two chambers in X belongs to X.

Theorem 4.4 (A. Borovik and S. Roberts [BR]) *Let W be a Coxeter group (not necessarily finite) and P a finite standard parabolic subgroup in W. Let $\mathcal{M}$ be a Coxeter matroid in W^P and $\mu : W \longrightarrow W^P$ the map which assigns to every $w \in W$ the w-maximal coset of $\mathcal{M}$. Then the following statements are true.*

(1) *The fibers $\mu^{-1}[A]$, $A \in W^P$, of the map μ are convex subsets of W.*

(2) *If two fibers $\mu^{-1}[A]$ and $\mu^{-1}[B]$ are adjacent (i.e. some chamber from $\mu^{-1}[A]$ is adjacent to a chamber in $\mu^{-1}[B]$) then their images A and B are symmetric with respect to some wall Σ of the Coxeter complex for W. Moreover, all common panels of $\mu^{-1}[A]$ and $\mu^{-1}[B]$ lie on Σ.*

An immediate translation of this theorem into the language of the convex geometry of the matroid polytope Δ yields the following result.

Theorem 4.5 *Let W be a finite Coxeter group and P a standard parabolic subgroup in W. Let $\mathcal{M}$ be a Coxeter matroid in W^P and $\mu : W \longrightarrow W^P$ the map which assigns to every $w \in W$ the w-maximal element of $\mathcal{M}$. Denote by*

$$\lambda(A) = \bigcup_{w \in \mu^{-1}[A]} w\overline{D},$$

for a coset $A \in W^P$, the union of the closures of the chambers corresponding to the elements w in the fiber $\mu^{-1}[A]$ of the map μ. Then the following statements are true.

(1) *The sets $\lambda(A)$, $A \in W^P$, are convex polyhedral cones in V.*

(2) *If two different cones $\lambda(A)$ and $\lambda(B)$ are adjacent, i.e. have a facet in common, then this facet lies on the mirror Σ of symmetry of the vertices δ_A and δ_B.*

5 Oriented matroids

Rather surprisingly the Symmetric Exchange Property gives a weakened version of the oriented matroid (chirotope) axioms. If χ is an alternating function from r-subsets of $[n]$ to $\{0, 1, -1\}$, then by [**BLSWZ**], p.128, χ is a chirotope if and only if

(B2′) For all $x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_r \in [n]$ such that

$$\chi(x_1, x_2, \ldots, x_r) \cdot \chi(y_1, y_2, \ldots, y_r) \neq 0,$$

there exists $i \in \{1, 2, \ldots, r\}$ such that

$$\chi(y_i, x_2, x_3, \ldots, x_r) \cdot \chi(y_1, y_2, \ldots, y_{i-1}, x_1, y_{i+1}, y_{i+2}, \ldots, y_r)$$
$$= \chi(x_1, x_2, \ldots, x_r) \cdot \chi(y_1, y_2, \ldots, y_r).$$

Equivalently, by [**BLSWZ**], p.138, χ is a chirotope if and only if both of the following hold:

(B1′) The set of r-subsets $\{x_1, x_2, \ldots, x_r\}$ of $[n]$ such that $\chi(x_1, x_2, \ldots, x_r) \neq 0$ is the collection of bases of an ordinary matroid on $[n]$,

and

(B2″) for any $x_1, x_2, \ldots, x_r, y_1, y_2 \in [n]$, if

$$\chi(y_1, x_2, x_3, \ldots, x_r) \cdot \chi(x_1, y_2, x_3, \ldots, x_r) \geq 0$$

and

$$\chi(y_2, x_2, x_3, \ldots, x_r) \cdot \chi(y_1, x_1, x_3, \ldots, x_r) \geq 0,$$

then

$$\chi(x_1, x_2, x_3, \ldots, x_r) \cdot \chi(y_1, y_2, x_3, \ldots, x_r) \geq 0.$$

Theorem 5.1 *Let χ be an alternating function from r-subsets of $[n]$ to $\{0, 1, -1\}$. Then χ is a chirotope if and only if the following axiom* (B2*) *is satisfied.*

> **(B2*)** *For all pairs of distinct subsets $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_r\}$ in $[n]$ such that*
>
> $$\chi(x_1, \ldots, x_r)\chi(y_1, \ldots, y_r) \neq 0$$
>
> *there exist $i, j \in \{1, 2, \ldots, r\}$ such that $x_i \notin Y, y_j \notin X$, and*
>
> $$\chi(x_1, \ldots, x_{i-1}, y_j, x_{i+1}, \ldots, x_r)\chi(y_1, \ldots, y_{j-1}, x_i, y_{j+1}, \ldots, y_r)$$
>
> *equals*
>
> $$\chi(x_1, \ldots, x_r)\chi(y_1, \ldots, y_r).$$

Proof. It is immediate that (B2′) implies (B2*), once we realize that by the alternating property, (B2′) is equivalent to

> *If $X = \{x_1, \ldots, x_r\}$ and $Y = \{y_1, \ldots, y_r\}$ are distinct r-subsets in $[n]$ such that*
>
> $$\chi(x_1, \ldots, x_r)\chi(y_1, \ldots, y_r) \neq 0$$
>
> *then for any given $i \in \{1, 2, \ldots, r\}$ with $x_i \notin Y$, there exists an index $j \in \{1, 2, \ldots, r\}$ such that $y_j \notin X$, and*
>
> $$\chi(x_1, \ldots, x_{i-1}, y_j, x_{i+1}, \ldots, x_r)\chi(y_1, \ldots, y_{j-1}, x_i, y_{j+1}, \ldots, y_r)$$
>
> *equals*
>
> $$\chi(x_1, \ldots, x_r)\chi(y_1, \ldots, y_r).$$

Conversely, we assume (B2*). Forgetting signs, we have the Fully Symmetric Exchange Property for ordinary matroids, which we proved was equivalent to the ordinary matroid axioms. Thus we have condition (B1′). Thus only condition (B2″) must be checked, which is the case where X has at most two elements not in Y, say $x_1, x_2 \notin Y$, and $y_1, y_2 \notin X$. We prove (B2″) by contrapositive. If $\chi(x_1, \ldots, x_r)\chi(y_1, \ldots, y_r) < 0$ then by (B2*), one of four possible exchanges ($i = 1$ or 2, and $j = 1$ or 2) must give a negative product of χ values. Each of these four possibilities contradicts the hypothesis of (B2″). □

The real reason why this works is that in the case X has only two elements not in Y, then specifying that x_1 must be exchanged is really the same as specifying that x_2 must be exchanged (by reversing the roles of the two resulting bases). Thus the apparently stronger (B2″) is equivalent to (B2*) in this case.

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