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Coxeter Theory: The Cognitive Aspects

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Abstract. Coxeter Theory is a fascinating example of a mathematical theory where the cognitive roots of mathematical thinking are directly evident. The paper contains an informal discussion, made from a mathematician's point of view, of possible cognitive aspects of the theory. We hope that it will be interesting to mathematicians and cognitive scientists alike.

Introduction

This paper is a mathematician's attempt to reflect on the explosive development of *mathematical cognition*, an emerging branch of neurophysiology which purports to locate structures and processes in the human brain responsible for mathematical thinking [13, 23]. So far the research efforts were concentrated mostly on the brain processes during quantification and counting; important as they are, these activities occupy a very low level in the hierarchy of mathematics. Not surprisingly, the remarkably achievements of cognitive scientists and neurophysiologists are mostly ignored by the mathematical community. The situation might change fairly soon, since conclusions drawn from the neurophysiological research could happen to be very attractive to policymakers in mathematical education, especially since neurophysiologists themselves do not shy away from making direct recommendations [14].

I refer the reader to the fascinating book *The Number Sense: How The Mind Creates Mathematics* by Stanislas Dehaene [23] for the first-hand account of the study into number sense and numerosity. In this paper, I am trying to bridge the gap between mathematics and mathematical cognition by pointing out to structures and processes of mathematics which, on one hand, are sufficiently non-trivial to be interesting to a mathematician, while, on the other hand, are deeply integrated into some basic structures of our mind and, hopefully, lie within the reach of a cognitive scientist. I will talk mostly about visual processing (and, in

An expanded version of a paper for the proceedings volume of the conference The Coxeter Legacy: Reflections and Projections, held at University of Toronto, 12–16 May 2004.

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particular, about our sense of symmetry), and about basic parsing procedures in language processing.

Remarkably, Coxeter Theory, that is, the theory of reflection groups and Coxeter groups as originated in seminal works of H. S. M. Coxeter [18, 19], provides a fascinating example of a mathematical theory where we occasionally have a glimpse of the inner working of our mind. As we shall soon see, the underlying cognitive mechanisms which make Coxeter Theory so natural and intuitive are deeply rooted in booth visual and verbal processing modules of our mind.

CONTENTS

Part 1 Mirrors and Reflections

1 The starting point: kaleidoscopes

One remarkable feature of Coxeter Theory is that its principal objects can be defined right on the spot in the most intuitive and elementary way. I give first an informal description:

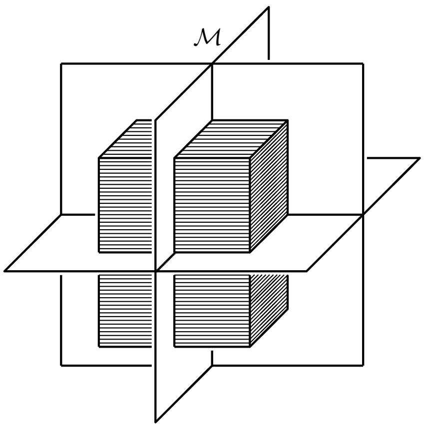
> imagine a few (semi-transparent) mirrors in ordinary three dimensional space. Mirrors (more precisely, their images) multiply by reflecting in each other, like in a kaleidoscope or a gallery of mirrors. A *closed system of mirrors* is what we see when we look into such a kaleidoscope.

As the reader of course knows, of special interest are systems of mirrors which generate only finitely many reflected images. One of the implications of Coxeter Theory is that such finite systems of mirrors are cornerstones of modern mathematics and lie at the heart of many mathematical theories.

Of course, the theory is actually concerned with the more general case of the n-dimensional Euclidean space, with mirrors being $(n-1)$-dimensional hyperplanes rather than two dimensional planes. To that end, we give a formal definition:

> a system of hyperplanes (mirrors) $\mathcal{M}$ in the Euclidean space $\mathbb{R}^n$ is called *closed* if, for any two mirrors M_1 and M_2 in $\mathcal{M}$, the mirror image of M_2 in M_1 also belongs to $\mathcal{M}$ (Figure 1).

Thus, the principal objects of Coxeter Theory can be described as *finite closed systems of mirrors*. In more general terms, the theory can be described as the geometry of multiple mirror images. This approach to Coxeter groups is well known and fully exploited, for example, in Chapter 5, §3 of Bourbaki's classical text [10][1], or in Vinberg's paper [58]. Of course, the definition of a finite closed mirror system is cryptomorphic (that is, equivalent but expressed in different language) to finite reflection groups and root systems. Its important property, however, is that it lends itself to a weakening and can be adapted to become a very efficient axiomatic of matroid theory, an important branch of combinatorics; see Section 13 for more detail.



The system $\mathcal{M}$ of mirrors of symmetry of a geometric body Δ is *closed*: the reflection of a mirror in another mirror is a mirror again. Notice that if Δ is compact then all mirrors intersect at a common point.

Figure 1 A closed system of mirrors.

So we have two aspects, expressed in two different mathematical languages, of the same mathematical theory (I will call it *Coxeter theory*). This is not an unusual thing in mathematics. What makes the case of mirror systems / Coxeter groups interesting is that a closer look at the corresponding mathematical languages reveals their cognitive (and even neurophysiological!) aspects, much more obviously than in the rest of mathematics. In particular, as we shall soon see, the mirror system / Coxeter group alternative precisely matches the great visual / verbal divide of mathematical cognition.

I wish to stress that, although the theory of Coxeter groups formally belongs to "higher" mathematics, the issues it raises are relevant to the teaching of mathematics at all levels,

[1] Arguably, one of the better books by Bourbaki; it even contains a drawing, which is an unexpected deviation from his usual style. See a very instructive discussion of the history of this volume by its main contributor, Pierre Cartier [47].

from elementary school to graduate studies. Indeed, I will be talking about such stuff as *geometric intuition*. I will also touch on the role of pictorial proofs and self-explanatory diagrams; some of them may look naïve, but, as I try to demonstrate, frequently lead deep into the heart of mathematics, see Section 6 for one of the more striking cases.

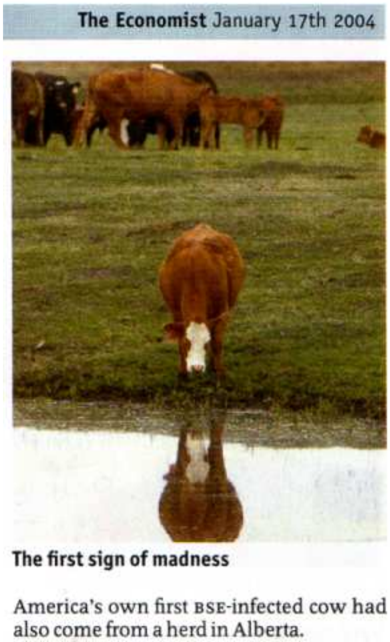


Figure 2 This illustration is taken from an article in *The Economist* about incidents of the Mad Cow Decease in North America. It clearly shows that the ability to recognize oneself in the mirror is equated, in popular culture, to the very self-awareness of a sentient being.

2 Image processing in humans

The mirror is, of course, one of the most powerful and evocative symbols of our culture; seeing oneself in a mirror is equated to the very self-awareness of a human being.[2] But the reason why the language of mirrors and reflections happens to be so useful in the exposition of mathematical theories lies not at a cultural but at a psychophysiological level.

How do people recognize mirror images? Tarr and Pinker [55] showed that recognition of planar mirror images is done by subconscious mental rotation of 180° about an appropriate axis. Remarkably, the brain computes the position of this axis!

This is how Pinker describes the effect of their simple experiment.

> So we showed ourselves [on a computer screen] the standard upright shape alternating with one of its mirror images, back and forth once a second. The perception of flipping was so obvious that we didn't bother to recruit volunteers to confirm it. When the shape alternated with its upright reflection, it seemed to pivot like a washing machine agitator. When it alternated with its upside-down reflection, it did backflips. When it alternated with

[2]Gregory [29] is a comprehensive survey of the cultural and psychological significance of mirrors.

> its sideways reflection, it swooped back and forth around the diagonal axis, and so on. *The brain finds the axis every time*. [43, pp. 282–283]

Interestingly, the brain is doing exactly the same with misorientated three-dimensional shapes, *provided they have the same orientation* and can be identified by a rotation [53]. The interested reader may wish to take any computer graphics package which allows animation and see it for himself.[3] It is really difficult to avoid the conclusion that the classical Euler's Theorem:

> If an orientation-preserving isometry of the affine Euclidean space $\mathbb{AR}^3$ has a fixed point then it is a rotation around an axis.

is hardwired into our brains.

The illusion of rotation disappears when the brain faces the problem of identification of three-dimensional mirror images of *opposite orientation*; of course, they can still be identified by an appropriate rotation, but, this time, in four-dimensional space. The environment which directed the evolution of our brain never provided our ancestors with four-dimensional experiences.

Human vision is a solution of an ill-posed inverse problem of recovering information about three-dimensional objects from two-dimensional projections on the retinas of the eyes. Pinker stresses that this problem is solvable only because of the many assumptions about the nature of the objects and the world in general built into the human brain or acquired from previous experiences.[4]

The algorithm of identification of three-dimensional shapes is only one of many modules in the immensely complex system of visual processing in humans. It is likely that various modules are implemented as particular patterns of connections between neurons. It is natural to assume that different modules developed at different stages of evolution of humans' ancestors [54]. The older ones were likely to be more primitive and, probably, involved relatively simple wiring diagrams. But since they had adaptive value, they were inherited and acted as constraints in the evolution of later additions to the system, new modules which happened to process the outputs of, and interact with, the pre-existent modules. At every stage, evolution led to the development of an algorithm for solving a very special and narrow problem.

The "flipping" algorithm for the recognition of mirror images of a flat object and the closely related (and possibly identical) "rotation" algorithm for the identification of misorientated three-dimensional objects provide rare cases when we can glimpse the inner workings of our mind. Observe, however, that the algorithms are solutions of relatively simple mathematical problems with a very rigid underlying mathematical structure, namely, the group of rotations of the three-dimensional Euclidean space. There is no analogue of Euler's Theorem for four-dimensional space!

The reader has possibly noticed that I prefer to use the term "algorithm" rather than "circuit", emphasizing the strong possibility that the same algorithms can be implemented

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[3] To reproduce Tarr's experiments, I was using PAINTSHOP PRO, with 3-dimensional images produced by XARA, both software packages picked from the cover disk of a computer magazine.

[4] Jody Azzouni [4, p. 125] made a subtle comment on pictorial proofs: they work only because we impose many assumptions on diagrams admissible as part of such proofs. As he put it,

> We can conveniently stipulate the properties of *circles* and take them as mechanically recognizable because there are no *ellipses* (for example) in the system. Introduce (arbitrary) *ellipses* and it becomes impossible to tell whether what we have drawn in front of us is a *circle* or an *ellipse*.

It is likely that his remark would not surprise cognitive psychologists; they believe that this is what our brain is doing anyway.

by different circuit arrangements if some of the arrangements became impossible as the result, say, of trauma, especially during the early stages of a child's development.[5]

Studies of compensatory developments are abundant in the literature. When I was looking for some recent studies, my colleague David Broomhead directed me to paper [28], a case study of a young woman who cannot make eye movements since birth but had surprisingly normal visual perception. It is surprising because the so-called saccadic movements of eyes are crucial for tracing the contours of objects. The woman compensates for the lack of eye movement by quick movements of her head which follow the usual patterns of saccadic movements. I quote the paper: "Her case suggests that saccadic movements, of the head or the eye, form the *optimal sampling method* for the brain". The italics are mine, since I find the choice of words very attractive: some aspect of the inner working of the brain is described as a mathematical procedure, which raises some really interesting metamathematical questions. I plan to write more on that elsewhere; meanwhile, I refer the interested reader to papers on mathematical models of eye movement [1, 11].

3 A small triumph of visualisation: Coxeter's proof of Euler's Theorem

If you need convincing that visualization might work in learning, teaching and doing mathematics, there is no better example than the proof of Euler's Theorem as it is given by Coxeter [20, p. 36]; I quote it *verbatim*. Remember that Coxeter's book was first published in 1948, hence was written for readers who were likely to have taken a standard course of Euclidean geometry and therefore had reasonably well-trained geometric imagination.

> In three dimensions, a congruent transformation that leaves a point **O** invariant is the product of at most three reflections: one to bring together the two x-axes, another for the y-axes, and a third (if necessary) for the z-axes.
>
> Since the product of three reflections is opposite, a direct transformation with an invariant point **O** can only be the product of reflections in *two* planes through **O**, i.e., a rotation.

I add just a few comments to facilitate the translation into modern mathematical language: a *congruent transformation* is an isometry; a *direct transformation* preserves the orientation, while an *opposite* one changes it. Coxeter refers to the fact that the product of two mirror reflections is a rotation about the line of intersection of mirrors. It is something that everyone has seen in a tri-fold dressing table mirror; the easiest way to prove the fact is to notice that the product of two reflections leaves invariant every point on the line of intersection of mirrors.[6]

We humans are blessed with a remarkable piece of mathematical software for image processing hardwired into our brains. Coxeter made the full use of it, and expected the reader to use it, in his lightning proof of Euler's Theorem. The perverse state of modern mathematics teaching is that "geometric intuition", the skill of solving geometric problems by looking at (simplified) two- and three-dimensional models is mostly expelled from classroom practice.

4 Mathematics: interiorization and reproduction

David Mumford [40, p. 199] paraphrased Davis and Hersh [21], to say that mathematics is

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[5] Compare Vandervert [57].

[6] We accept that the reader has every right to insist that the best way to prove Euler's Theorem is by reduction to algebra and eigenvalues of a three dimensional orthogonal matrix. But is that simpler than Coxeter's proof?

the study of mental objects with reproducible properties.[7]

Thus learning mathematics has at least two intertwined aspects:

- Interiorization of other people's mental objects.
- The development of reproduction techniques for your own mental objects.

There is a natural hierarchy of reproduction methods. A partial list includes: proof; algorithm; symbolic and graphic expression. I wish to clarify that reproduction is more than communication: you have to be able to reproduce your own mental work for yourself.

Interiorization is less frequently discussed; for our purposes, we mention only that it includes visualization of abstract concepts; transformation of formal conventions into psychologically acceptable "rules of the game"; development of subconscious "parsing rules" for the processing of strings of symbols (most importantly, for reading mathematical formulae). At a more mundane level, you cannot learn an advanced technique of symbolic manipulation without first polishing your skills in more routine computations to the level of almost automatic perfection. Interiorization is more than understanding; to handle mathematical objects, one has to imprint at least some of their functions at the subconscious level of one's mind. My use of the term "interiorization" is slightly different from the understanding of this word, say, by Weller et al. [59]. I put emphasis on subconscious, neurophysiological components of the process.[8]

Some mathematical activities are of synthetic nature and can be used as means of both interiorization and reproduction. A really remarkable one is the generation and discussion of examples. Really useful examples can be loosely divided into two groups: "typical", generic examples of the theory, or, on the contrary, "simplest possible", almost degenerate examples, which emphasize the limitations and the logical structure of the theory. Of course, one of the attractive features of the Coxeter Theory is that it is saturated by beautiful examples of both kinds; I discuss some "exceptional" cases in Section 6.

Proof, being the highest level of reproduction activity, has an important interiorization aspect: as Yuri Manin stresses, a proof becomes such only after it is *accepted* [37, pp. 53–54]. Manin describes the act of acceptance as a social act; however, the importance of its personal, psychological component could hardly be overestimated.

Visualization is one of the most powerful interiorization techniques. It anchors mathematical concepts and ideas into one of the most powerful parts of our brain, the visual processing module. Returning to Coxeter Theory, I want to point out that finite reflection groups allow an approach to their study based on a systematic reduction of complex geometric configurations to much simpler two- and three-dimensional special cases. Mathematically it is expressed by Coxeter's theorem:

a finite reflection group is a Coxeter group,

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[7] Mumford continues:

> I love this definition because it doesn't try to limit mathematics to what has been called mathematics in the past but really attempts to say why certain communications are classified as math, others as science, others as art, others as gossip. Thus reproducible properties of the physical world are science whereas reproducible mental objects are math. Art lives on the mental plane (the real painting is not the set of dry pigments on the canvas nor is a symphony the sequence of sound waves that convey it to our ear) but, as the post-modernists insist, is reinterpreted in new contexts by each appreciator. As for gossip, which includes the vast majority of our thoughts, its essence is its relation to a unique local part of time and space.

[8] Meanwhile, I am happy to borrow from [59] the terms *encapsulation* and *de-encapsulation*, to stand for the conversion of a mathematical procedure, a learnt sequence of action, into an object.

that is, all relations between elements are consequences of relations between *pairs* of generating reflections. But a pair of mirrors in the n-dimensional Euclidean space is no more sophisticated a configuration than a pair of lines on the plane, and all the properties of the former can be deduced from that of the latter. *This provides a metamathematical explanation why visualization is so effective in the theory of finite reflection groups.*

5 How to draw an icosahedron on a blackboard

My understanding of visualization as an interiorization technique leads me to believe that drawing pictures is an important way of facilitating mathematical work. This means that pictures have to be treated as mathematical objects, and, consequently, be *reproducible*.

I have to emphasize the difference between *drawings* or *sketches* which are supposed to be reproduced by the reader or student, and more technically sophisticated illustrative material (I will call it *illustrations*), especially computer-generated images designed for visualization of complex mathematical objects. It would be foolish to impose restrictions on the technical perfection of illustrations. [9] I believe that *drawings* should be intentionally made very simple, almost primitive. Mathematical pictures represent *mental* objects, not the real world! In words of one of the leading geometers of our time, William Thurston, people

> do not have a very good built-in facility for *inverse vision*, that is, turning an internal spatial understanding back into a two-dimensional image. Consequently, mathematicians usually have fewer and poorer figures in their paper and books than in their heads. [56, p. 164]

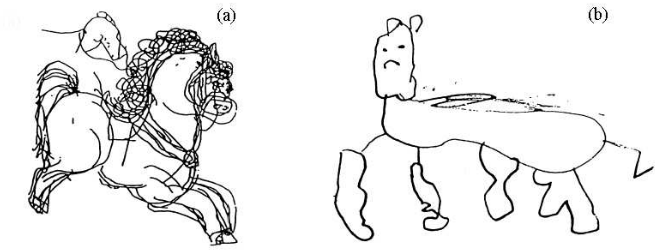


Figure 3 This pictures from [48] (reproduced with permission) illustrate the concept of "inverse vision" as introduced by Thurston. The picture on the left (Selfe [46]) is drawn from memory by a mentally retarded three-and-a-half year old autistic child. Picture (b) is a representative drawing of a normal child, at age four years and two months. It is obvious that a normal child draws not a horse, but a concept of a horse.

Mathematical pictures should not instil an inferiority complex in the reader who has not attempted to draw anything since his or her halcyon days at the elementary school; they should act as an invitation to the reader to express his or her own mental images.

[9] However, one should be aware of the danger of excessive details; as William Thurston stresses, words, logic and detailed pictures rattling around can inhibit intuition and associations. [56, p. 165]

Figure 4 illustrates possibly the most effective way of drawing an icosahedron, so simple that it is accessible to the student with very modest drawing skills.[10] First we mark symmetrically positioned segments in an alternating fashion on the faces of the cube (left), and then connect the endpoints (right). The drawing actually provides a proof of the existence of the icosahedron: varying the lengths of segments on the left cube, it is easy to see from the continuity principles, that, at certain length of the segments, all edges of the inscribed polytope on the right become equal. Moreover, this construction helps to prove that the group of symmetries of the resulting icosahedron is as big as it should be.

Figure 4 works as a proof because it is produced by "inverse vision". To draw it, you have to run, in your head, the procedure for construction of the icosahedron. And, of course, the continuity principles used are self-evident – they are part of the same mechanisms of perception of motion which glue, in our mind, cinema's 24 frames a second in a continuous motion.

I hope that you agree with me that Figure 4 deserves to be treated as a mathematical statement.

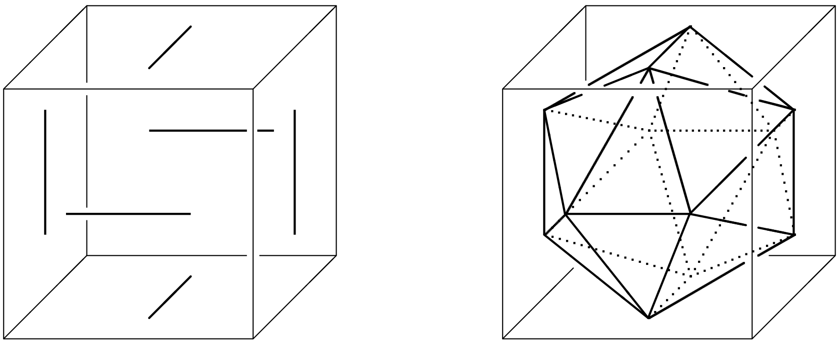


Figure 4 A self-evident construction of an icosahedron.

Of course, construction of the icosahedron is the same as construction of the finite reflection group H_3; it can be done by means of linear algebra – which leads to rather nasty calculations, or by means of representation theory – which requires some knowledge of representation theory. It also can be done by quaternions – which is nice and beautiful, but requires the knowledge of quaternions. The graphical construction is the simplest; using computer jargon, it is WYSIWYG ("What You See Is What You Get") mode of doing mathematics, which deserved to be used at every opportunity.

6 Self-explanatory diagrams

Self-explanatory diagrams are virtually expunged from modern mathematics. I believe they might be useful, maybe not as formal tools for use in proofs, etc., but as means of metamathematical discussion of the structure and interrelations of mathematical theories.

[10]This construction of the icosahedron is adapted from the method of H. M. Taylor [33, pp. 491–492]. John Stillwell has kindly pointed out the it goes back to Piero della Francesca in his unpublished manuscript *Libellus de quinque corporibus regolaribus* from around 1480.

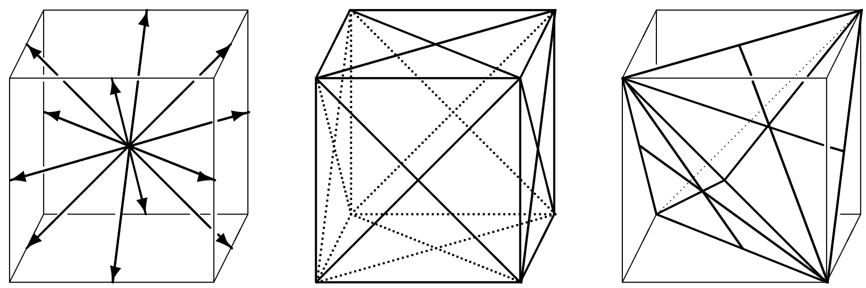


Figure 5 The mirror system of type D_3 is the same as the mirror system of type A_3.

Figure 5 (taken from [8]) is one example: the isomorphism of the root systems D_3 (shown on the left inscribed into the unit cube $[-1, 1]^3$) and A_3 is not immediately obvious, but the corresponding mirror systems coincide most obviously. The mirror system D_3 (the system of mirrors of symmetry of the cube) is shown in the middle by tracing the intersections of mirrors with the surface of the cube, and, on the right, by intersections with the surface of the tetrahedron inscribed in the cube. Comparing the last two pictures we see that the mirror system of type D_3 is isomorphic to the mirror system of the regular tetrahedron, that is, to the system of type A_3.

As we shall soon see, this isomorphism has far-reaching implications.

Indeed, at the level of complex Lie groups the isomorphism $D_3 \simeq A_3$ becomes the rather mysterious isomorphism between the 6-dimensional orthogonal group $\mathrm{SO}_6(\mathbb{C})$ and $\frac{1}{2}\mathrm{SL}_4(\mathbb{C})$, the factor group of the 4-dimensional special linear group $\mathrm{SL}_4(\mathbb{C})$ by the group of scalar matrices with diagonal entries ± 1 (or, if you prefer to work with spinor groups, between $D_3(\mathbb{C}) = \mathrm{Spin}_6(\mathbb{C})$ and $A_3(\mathbb{C}) = \mathrm{SL}_4(\mathbb{C})$).

This is not yet the end of the story. The compact form of $\mathrm{SL}_4(\mathbb{C})$ is SU_4, hence the embedding

$$\mathrm{SU}_4 \hookrightarrow \mathrm{Spin}_6(\mathbb{C})$$

features prominently in the representation theory of SU_4, and hence in the SU_4-symmetry formalism of theoretical physics.

But the underlying reason for the isomorphisms is still ridiculously elementary: retains all the audacity of Keplerian reductionism: the tetrahedron can be inscribed into the cube, just compare with Figure 6.

Because of their truly fundamental role in mathematics, even the simplest diagrams concerning finite reflection groups (or finite mirror systems, or root systems—the languages are equivalent) have interpretations of cosmological proportions. Figure 7 is even more instructive. It is a classical case of the *simplest possible example* as discussed in Section 4. For example, it is the simplest rank 2 root system, or the simplest root system with a non-trivial graph automorphism; the latter, as we shall see in a minute, has really significant implications.

Figure 7 also demonstrates that the root system $D_2 = \{\pm\epsilon_1 \pm \epsilon_2\}$ is isomorphic to $A_1 \oplus A_1$. At the level of Lie groups, this isomorphism plays an important role in the description of the structure of 4-dimensional space-time of special relativity, namely,

Figure 6 A fragment of one of the famous engravings from Kepler's *Mysterium Cosmographicum*.

it yields the structure of the Minkowski group, that is, the group of isometries of the 4-dimensional space-time of special relativity theory with the metric given by quadratic form

$$x^2 + y^2 + z^2 - t^2.$$

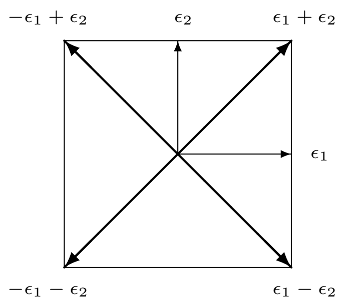


Figure 7 The isomorphism of root systems $D_2 = \{ \pm\epsilon_1 \pm \epsilon_2 \}$ and $A_1 \oplus A_1$.

Indeed, the isomorphism of root systems $D_2 \simeq A_1 \oplus A_1$ leads to the isomorphisms

$$\mathrm{Spin}_4(\mathbb{C}) \simeq \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$$

and

$$\mathrm{SO}_4(\mathbb{C}) \simeq \mathrm{SL}_2(\mathbb{C}) \otimes \mathrm{SL}_2(\mathbb{C})$$

(the tensor product of two copies of $\mathrm{SL}_2(\mathbb{C})$, each acting on its canonical 2-dimensional space $\mathbb{C}^2$). The Minkowski group is a real form of $\mathrm{SO}_4(\mathbb{C})$). Hence it is the group of fixed points of some involutory automorphism τ of $\mathrm{SO}_4(\mathbb{C})$). What is this automorphism τ? Let us look again at the quadratic form

$$x^2 + y^2 + z^2 - t^2;$$

it ought to be a real form of the complex quadratic form

$$z_1^2 + z_2^2 + z_3^2 + z_4^2,$$

but lost the symmetric pattern of coefficients. One can see that this means that τ swaps the two copies of $\mathrm{SL}_2(\mathbb{C})$ in $\mathrm{SL}_2(\mathbb{C}) \otimes \mathrm{SL}_2(\mathbb{C})$ and therefore has to be the symmetry between the two diagonals of the square in Figure 7. Being an involution, τ fixes pointwise the "diagonal" subgroup in $\mathrm{SL}_2(\mathbb{C}) \otimes \mathrm{SL}_2(\mathbb{C})$ isomorphic to $\mathrm{PSL}_2(\mathbb{C})$. (It is $\mathrm{PSL}_2(\mathbb{C})$ rather than $\mathrm{SL}_2(\mathbb{C})$ because its center $\langle -\mathrm{Id} \otimes -\mathrm{Id} \rangle$ is killed in the tensor product.) Hence the Minkowski group is isomorphic to $\mathrm{PSL}_2(\mathbb{C})$.

Three cheers for Kepler!

Part 2 Words and Brackets

7 Parsing

So far I have emphasized the role of visualization in mathematics, and its power of persuasion. Here I will try to relate the visual and symbolic aspects of mathematics, and touch on the limitations of visualization.

Indeed, visualization works perfectly well in the naive geometric theory of finite reflection groups, as long as you do not venture into the more general and stunningly beautiful theory of (infinite) Coxeter groups. Being basic and truly fundamental mathematical objects, Coxeter groups also provide an example of a theory where the links of mathematical teaching/learning to cognitive psychology lie exposed. Besides the power of geometric interpretation and visualization, the theory of Coxeter groups very much relies on the manipulation of words in canonical generators (chains of consecutive reflections, in the case of reflection groups) and provides one of the best examples of the effectiveness of the language metaphor in mathematics.[11]

Of course, one can be tempted to try to link the psychology of symbolic manipulation in mathematics with the conjecture, first formulated by Chomsky and vigorously promoted by Steven Pinker [44], that humans have an innate facility for parsing human language. Basically, parsing is the introduction of structure into strings of symbols (phonems, letters, etc.) We are parsing everything we read or hear; here is an example from Pinker's book [44, pp. 203–205]:

Remarkable is the rapidity of the motion of the wing of the hummingbird.

To make the sense of the phrase, we have to mentally bracket the linked words, making something like

[Remarkable is [the rapidity of [the motion of [the wing of [the hummingbird]]]]].

A sentence might have a different bracket pattern, just compare

[Remarkable is [the rapidity of [the motion]]]

and

[[The rapidity that [the motion] has] is remarkable].

---

[11] For a further development of the language metaphor, see, for example, the book [25] which discusses applications of the theory of formal languages to group theory.

Some patterns are harder to deal with than others:

[[The rapidity that [the motion that [the wing] has] has] is remarkable],

while some come close to incomprehensible, even if the sentence conveys the same message:

[[The rapidity that [the motion that [the wing that [the hummingbird] has] has] has] is remarkable].

Different human languages have different grammars, resulting in different parsing patterns. The grammar is not innate; Pinker emphasizes that what is innate is the human capacity to generate parsing rules. Generation of parsing patterns is a part of language learning (and infants are extremely efficient in it). Also, it is a part of the interiorization of mental objects of mathematics, especially when they are represented by strings of symbols.

Cognitive scientists are very much attracted to case studies of "idiots savants", autistic persons with a disproportionate ability, in comparison with their low general IQ, to handle arithmetic or calendrical calculations. As Snyder and Mitchell formulated it [48],

> . . . savant skills for integer arithmetic . . . arise from an ability to access some mental process which is common to us all, but which is not readily accessible to normal individuals.

What are these "hidden" processes? In one of the extreme cases (mentioned by Butterworth [14]), a severely autistic young man was unable to understand speech but could handle factors and primes in numbers. This suggests that certain mathematical actions are related not so much to language itself, but to the parsing facility, one of the components of the language system; an autistic person might have difficulty in handling language for reasons unrelated to his parsing ability, for example for his incapacity to recognize the source of speech communication as another person. But, in order to achieve such feats as "doubling 8 388 628 up to 24 times to obtain 140 737 488 355 328 in several seconds" [48, p. 589], an autistic person still has to be able to input into his brain the numbers given, inevitably, as strings of phonemes or digits.

I dare to suggest that the parsing mechanisms of the human brain are the key to the understanding of low-level arithmetic and formula processing.

Moving several levels up the hierarchy of mathematical processes, we have a fascinating idea in the theory of automatic theorem proving: *rippling*, a formalization of a common way of mathematical reasoning where "formulae are manipulated in a way that increases their similarities by incrementally decreasing their differences" [12, p. 13]. This is facilitated by differentiating the formula into parts which have to be preserved and parts which have to be changed. Again, we see that in order understand how humans use rippling in mathematical thinking (and whether they actually use it), we have to understand how our brain parses mathematical formulae.

8 Number sense and grammar

I turn to another remarkable story from cognitive psychology, which links mechanisms of language processing to mastering arithmetic.

When infants learn to speak (in English) and count, there is a distinctive period, of 5-6 months, in their development, when they know the words *one, two, three, four*, but can correctly apply only the numeral "one", when talking about a single object; they apply words "two, three, four", apparently at random, to any collection of more than one object. Susan Carey [15] calls the children at this stage *one-knowers*. The most natural explanation is that they react to the formal grammatical structures of the adults' speech: *one doll*, but *two dolls, three dolls*. At the next stage of development, they suddenly start using the numerals *two, three, four, five* correctly. Chinese and Japanese children become one-knowers a few months later – because the grammar of their languages has no specific markers for singular or plural in nouns, verbs, and adjectives.

When the native language is Russian, the "one-knower" stage is replaced by "two-three-four knower" stage, when children can differentiate between three categories of quantities: single object sets, the sets of two, three of four objects (without further differentiation between, say, two or three objects), and sets with five or more objects. This is happening because the morphological differentiation of plural forms in the Russian language goes further than in English.

Well, when I heard about special plural forms of two, three or four nouns in a lecture by Susan Carey at the *Mathematical Knowledge 2004* conference in Cambridge, I was mildly amused because it made no sense to me, a native Russian speaker. Still, I started to write on note paper:

one doll	одна кукл**А**
two doll**S**	две кукл**Ы**
three doll**S**	три кукл**Ы**
four doll**S**	четыре кукл**Ы**
five doll**S**	пять кук**ОЛ**
⋮	⋮
ten doll**S**	десять кук**ОЛ**

I was startled: yes, Susan Carey was right! I was using, all my life, the morphological rules for forming plurals without ever paying any attention to them, subconsciously. But, apparently, an infant's brain is tuned exactly at picking the rules: it is easier for the child to associate the number of objects with the morphological marker in the noun signifying the object than with the word *one* or *two*.[12] *In learning numbers, the grammar precedes words!*

9 Palindromes and mirrors

To demonstrate the role of parsing and other word processing mechanisms in doing mathematics, let us briefly describe Coxeter groups in terms of words, intentionally using as low level "non-mathematical" terminology as possible..

We work with an alphabet $\mathbb{A}$ consisting of finitely many letters, which we denote $a, b, \ldots$, etc. A *word* is any finite sequence of letters, possibly empty (we denote the empty word ϵ). Notice that we have infinitely many words. To impose an algebraic structure onto the amorphous mass of words, we proclaim that some of them are equivalent (or synonymous) to other words; we shall denote the equivalence of words V and W by writing $V \equiv W$. We demand that concatenation of words preserves equivalence: if $U \equiv V$ then $UW \equiv VW$ and $WU \equiv WV$: if *mail* is the same as *post* then *mailroom* is the same as *postroom*. We denote the language defined by the equivalence relation $\equiv$ by $\mathcal{L}_{\equiv}$.

So far all that was just the proverbial "general nonsense". It is remarkable how little we have to add in order to create an extremely rigid, crystal-like structure of a Coxeter group. To that end, we say that a word is *reduced* if it is not equivalent to any shorter word. Now we introduce just two axioms which define *Coxeter languages*:

---

[12] See a detailed discussion of plurality marking in Sarnecka et al. [45].

DELETION PROPERTY: If a word $W = a_1 \cdots a_k$ is *not* reduced, then W is equivalent to a word

$$a_1 \cdots a_{i-1} a_{i+1} \cdots a_{j-1} a_{j+1} \cdots a_k$$

obtained from $W = a_1 \cdots a_k$ by deleting some two letters a_i and a_j.

(Of course, it may happen that the new word is still not reduced, in which case the process continues in the same fashion, two letters at a time.)

REFLEXIVITY: The words like aa obtained by doubling a letter are not reduced (hence equivalent to the empty word, by the Deletion Property); *aardvark* is not a reduced word.

Actually, a Coxeter language is exactly a Coxeter group, but I intentionally ignore this crucial fact and formulate everything in terms of words and languages.

I can now give a (rather straightforward and simple) reformulation of a classical theorem of 20th century algebra, due to Coxeter and Tits. My formulation is a bit of a caricature and invented specifically for the purposes of the present paper.

To emphasize the language aspects, I make a *palindrome*, that is, a word that reads the same backwards as forwards, the central object of the theory. When talking about Coxeter languages, I will make an extra technical assumption that *palindromes are reduced and non-empty*. Now the Coxeter-Tits Theorem becomes a theorem about representations of palindromes by mirrors.

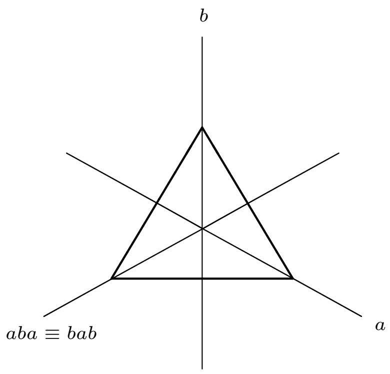


Figure 8 The Palindrome Representation Theorem: The three mirrors of symmetry of the equilateral triangle correspond to the palindromes a, b and aba. Together with the equivalences $aa \equiv bb \equiv \epsilon$ (the empty word), the equivalence $aba \equiv bab$ warrants that the corresponding Coxeter language does not contain any other palindromes.

The Palindrome Representation Theorem. Assume that a Coxeter language $\mathcal{L}_\equiv$ contains, up to equivalence, only finitely many palindromes.[13] Then:

[13] Without this finiteness assumption, the Palindrome Representation Theorem is still true if we accept mirrors in non-Euclidean spaces.

- There exists a finite closed system $\mathcal{M}$ of mirrors in a finite-dimensional Euclidean space $\mathbb{R}^n$ such that the mirrors in $\mathcal{M}$ are in one-to-one correspondence with the classes of equivalence of palindromes.
- Moreover, if M_1 and M_2 are mirrors and P_1, P_2 their palindromes, then the palindrome associated with the reflected image of the mirror M_1 in the mirror M_2 is equivalent to $P_2P_1P_2$.
- Finally, every closed finite system of mirrors in the Euclidean space $\mathbb{R}^n$ can be obtained in that way from the system of palindromes in an appropriate Coxeter language.

The interested reader may find all the necessary ingredients of a proof of this result in Chapters 5 and 7 of [9]. It involves, at some point, the following identity [8, Exercise 11.8]:

$$a_1 \cdots a_l \equiv a_l^{a_{l-1}\cdots a_1} \cdot a_{l-1}^{a_{l-2}\cdots a_1} \cdots a_2^{a_1} \cdot a_1,$$

where the group conjugation $b^{a_k \cdots a_1}$ can be viewed simply as an abbreviation for the symmetric (or palindromical) expression

$$a_1 \cdots a_k \cdot b \cdot a_k \cdots a_1;$$

The identity expresses an arbitrary word as the concatenation of palindromical words; its proof consists of rearrangement of brackets and cancellation of doubled letters a_ia_i whenever they appear. Proofs like that is one of the many reasons why, in order to master the theory of Coxeter groups expressed in a "linguistic" manner, the novice reader has to develop the ability to manipulate the imaginary mental brackets with a rapidity comparable only with the remarkable rapidity that the motion of the wing of the hummingbird has.

My "palindrome" formulation of the Coxeter-Tits Theorem is one of many manifestations of *cryptomorphism*, the remarkable capacity of mathematical concepts and facts for translation from one mathematical language to another. I emphasize that, in this paper, I adopted a "local", "microscopic" viewpoint. Although the "palindrome theory" is of little "global" value for mathematics in general, it is sufficiently amusing and demonstrates some interesting "local" features of mathematics.

I stress again that I invented the palindrome formulation of the Representation Theorem specifically for the present paper. When afterwards I made a standard search on Google and MathSciNet, I was pleased to discover that my formulation appeared to be new.

I was also pleasantly surprised to find more than a hundred papers on palindromes produced by computer scientists. The set of all palindromic words in a given alphabet is one of the simplest examples of a language which can be generated only by a device with some kind of memory, say, with a stack or push-down storage which works on the principle "last come – first go", like bullets in a handgun clip. It makes palindromes a very attractive test problem in the study of the complexity of word processing, for example, for comparing the two concepts of complexity: space-complexity, measured by the amount of memory required, and time-complexity. The difference between the two complexities is deeply philosophical: we can re-use space, but, unfortunately, cannot re-use time. I was particularly fascinated to learn that palindromes are recognizable by Turing machines working within sublogarithmic space constraints [52]. Hence, in this particular problem it is possible to overwrite and re-use the memory.

Maybe, it is exactly the necessity to engage – and re-use – the memory that turns palindromes into such popular and addictive brainteasers.

10 Parsing, continued: do brackets matter?

The balance of interiorization/reproduction is crucial for any serious discussion of what is actually happening in teaching and learning mathematics, and it is very worrying that this cognitive core is so frequently missing from the professional discourse on mathematical education. This is especially true for the discussion of merits of computer-assisted learning of mathematics, where the use of technology changed the cognitive content of standard elementary routines which for centuries served as building blocks for the learning of mathematics.

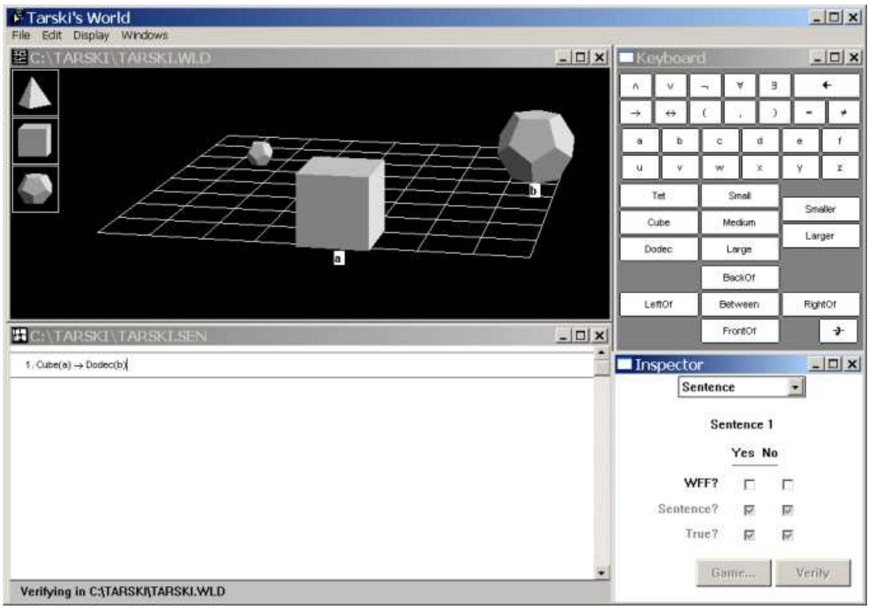


Figure 9 A screen shot of TARSKI'S WORLD.

And here is a small case study, concerned mostly with the basic parenthesizing of the “wing of the hummingbird” sort, as in Section 7. For some years I taught courses in mathematical logic based on two well-known software packages: SYMLOG [42] and TARSKI'S WORLD [5] (reviews: [7, 34]). SYMLOG used a DOS command line interface which was extremely poor even by the standards of its time, while TARSKI'S WORLD very successfully exploited the graphic user interface of Apple and Windows for the visualization of one of the key concepts of logic, a model for a set of formulae, see [6] for the discussion of the underlying philosophy. Also, TARSKI'S WORLD made a very clever use of games for explaining another key concept, the validity of a formula in an interpretation (although the range of interpretations was limited [34]). However, when it came to a written test, students taught with SYMLOG made virtually no errors in composition of logical formulae, while those taught with TARSKI'S WORLD very obviously struggled with this basic task. The reason was easy to find: SYMLOG's very unforgiving interface required retyping the whole formula if its syntax had not been recognized, while TARSKI'S WORLD's user-friendly formula editor automatically inserted matching brackets. Although TARSKI'S WORLD's students had no difficulty with rather tricky logic problems when they used a computer, their inability to handle formulae without a computer was alarming. Indeed, in mathematics, the ability to reproduce your mental work has to be media-independent. Relieving the students of a repetitive and seemingly mindless task led them to lose a chance to develop an essential skill.

It is appropriate to mention that, besides visualization, there is another mode of interiorization, namely *verbalization*. Indeed, we much better understand those things which we can describe in words. In naive terms, typing a command is like saying a sentence, while clicking a mouse is equivalent to pointing a finger in conversation. The reader would probably agree that, when teaching mathematics, we have to make our students speak. The tasks of opening and closing matching pairs of brackets, however dull and mundane they are, *activate* the deeply rooted neural mechanisms for generation of parsing rules, and are crucial for the interiorization of symbolic mathematical techniques.

11 The mathematics of bracketing and Catalan numbers

We have not begun to understand the relationship
between combinatorics and conceptual mathematics.

Jean Dieudonné [24]

The parsing examples we have considered so far were of a special kind, *binary parenthesizing*; I do not want to venture into anything more sophisticated because even placing parentheses in an expression made by repeated use of a binary operation, like

$$a + b + c + d$$

is already an immensely rich mathematical procedure. In various disguises, it appears throughout the entire realm of mathematics. There is no better example than Richard Stanley's famous collection of 66 problems on Catalan numbers [50, Exercise 6.19, pp. 219–229][14]. I quote a couple of examples.

The number of various ways to parenthesize the sum of $n + 1$ numbers,

$$a_1 + a_2 + \cdots + a_n + a_{n+1}$$

is called the *n-th Catalan number* and is denoted C_n; it can be shown that

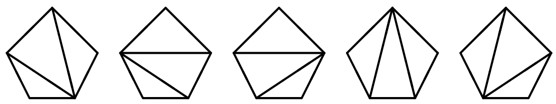
$$C_n = \frac{1}{n+1} \cdot \binom{2n}{n}.$$

For example, when $n = 3$, we have 5 ways to place the brackets in $a + b + c + d$:

$$a+(b+(c+d)),\ a+((b+c)+d),\ (a+(b+c))+d,\ (a+b)+(c+d),\ ((a+b)+c)+d$$

(following the usual convention, I skip the outmost pair of brackets).

Remarkably, when you count the ways to triangulate a convex $(n + 2)$-gon by $n - 1$ diagonals which touch each other only at their endpoints, you come to the same result:

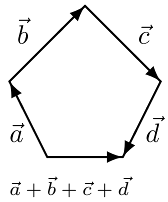


[14] Solutions can be found on Internet [51].

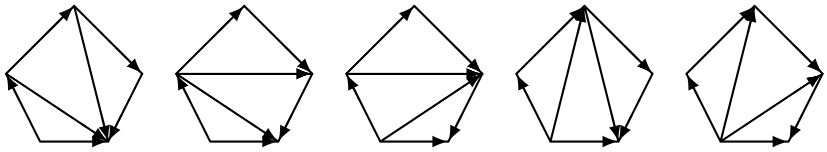
The mysterious coincidence is resolved as soon as we treat drawing diagonals as taking the sums of $n+1$ vectors

$$\vec{a}+\vec{b}+\vec{c}+\vec{d}$$

going along the $n+1$ sides of the $(n+2)$-gon , with the last side (the base of the polygon) representing the sum:

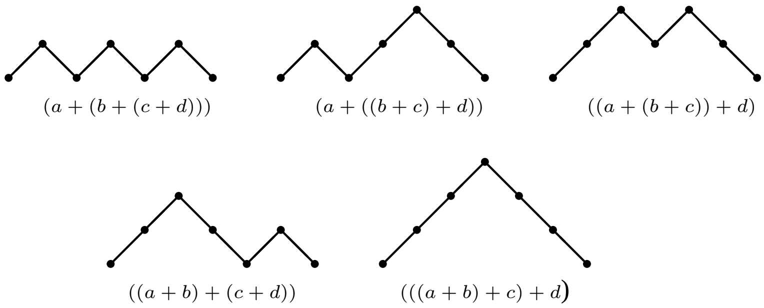


Now the one-to-one correspondence between parenthesizing the vector sum and drawing the diagonals becomes self-evident:



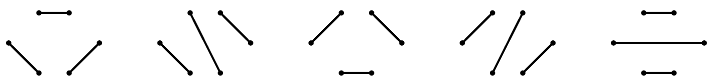
$\vec{a}+(\vec{b}+(\vec{c}+\vec{d}))$ $\vec{a}+((\vec{b}+\vec{c})+\vec{d})$ $(\vec{a}+(\vec{b}+\vec{c}))+\vec{d}$ $(\vec{a}+\vec{b})+(\vec{c}+\vec{d})$ $((\vec{a}+\vec{b})+\vec{c})+\vec{d}$

As a teaser to the reader I give another class of combinatorial objects which are also counted by Catalan numbers. Take some graph paper with a square grid, and assume that the unit (smallest) squares have length 1. A *Dyck path* is a path in the grid with steps $(1,1)$ and $(1,-1)$. I claim that the number of Dyck paths from $(0,0)$ to $(2n,0)$ which never fall below the coordinate x-axis $y=0$ is, again, the Catalan number C_n. I give here the list of such paths for $n=3$, arranged in a natural one-to-one correspondence with the patterns of parentheses in $a+b+c+d$:



Can you describe the rule? Notice that I added, for your convenience, the exterior all-embracing pairs of parenthesis, which are usually omitted in algebraic expressions.

One more example is concerned with n nonintersecting chords joining $2n$ points on the circle (or on an oval):



Again, there are

$$C_n = \frac{1}{n+1} \cdot \binom{2n}{n}$$

different ways to draw the chords. Can you find a one-to-one correspondence between the 5 chord diagrams and the 5 ways to parenthesize the sum $a + b + c + d$?[15]

Richard Stanley [50, pp. 219–229] has a list of 66 similar problems, each, of course, having Catalan numbers as the answer! He makes a wry comment that, ideally, the best way to solve the problems is to construct directly the one-to-one correspondences between the 66 sets involved, $66 \cdot 65 = 4290$ bijections in all!

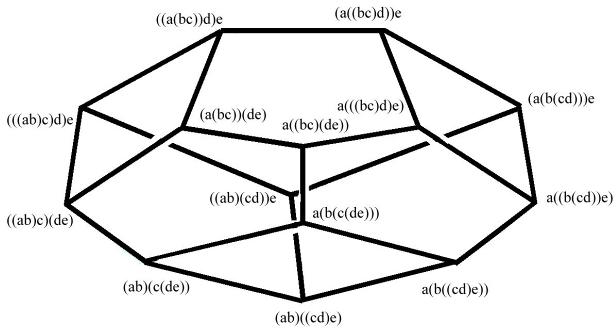


Figure 10 Stashef's associahedron: binary parenthesizings of n symbols can be arranged as vertices of a convex $(n-2)$-gon, with two vertices connected by an edge if the corresponding parenthesizings differ by position of just one pair of brackets.

This is still not the end of the story: the striking influence of a seemingly mundane structure, grammatically correct parenthesizing, can be traced all the way to the most sophisticated and advanced areas of modern mathematics research. A brief glance at Stashef's associahedra (Figure 10) suggests that they live in the immediate vicinity of the Coxeter Theory.[16] Actually, generalised associahedra can be defined for any finite Coxeter group (Stashef's associahedra being associated, of course, with the symmetric group Sym_n viewed as the Coxeter group of type A_{n-1}); for some recent results see, for example, [26].

12 The mystery of Hipparchus

It appears that the importance of parsing was appreciated by mathematicians and philosophers from truly ancient times. The following fragment from Plutarch, a famous Greek biographer and philosopher of 2nd century A.D., remained a mystery for centuries:

[15] I can give you a hint: the chords link opening parentheses "(" with the plus symbol "+" last applied in the sum enclosed by the matching closing parenthesis.

[16] An elemntary construction of associahedra can be found in Loday [36].

> Chrysippus says that the number of compound propositions that can be made from only ten simple propositions exceeds a million. (Hipparchus, to be sure, refuted this by showing that on the affirmative side there are 103,049 compound statements, and on the negative side 310,952.)

Here Plutarch refers to two prominent thinkers of Classical Greece: the philosopher Chrysippus (c. 280 B.C.–207 B.C.) and the astronomer Hipparchus (c. 190 B.C.–after 127 B.C.). Only in 1994 did David Hough notice that 103,049 is the number of arbitrary (non-binary) parenthesizings of 10 symbols, that is, the number of all possible expressions like

$$(xxxx)((xx)(xx)xx).$$

This suggests that, for Chrysippus and Plutarch, "compound" propositions were built from "simple" propositions simply by bracketing.

The mathematics and history of Hipparchus' number is discussed in detail in a paper by Richard Stanley [49]. The number of parenthesizings of n symbols is known as the *Schröder number* $s(n)$; the first 11 values of Schröder numbers are

$$1,\ 1,\ 3,\ 11,\ 45,\ 197,\ 903,\ 4279,\ 20793,\ 103049,\ 518859.$$

In 1998, Laurent Habsieger, Maxim Kazarian and Sergei Lando [30] suggested a very plausible explanation of the second Hipparchus number, of compound statements on "negative side". They observe that

$$\frac{s(10)+s(11)}{2} = 310,954$$

and, assuming a slight arithmetic or copying error in Plutarch's text, suggest to interpret the compound statements on the "negative side" as parenthesizings of expressions

$$\textsc{Not}\, x_1x_2\cdots x_{10}$$

under the following convention: the negation NOT is applied to all the simple propositions included in the first brackets that include NOT. That means that parenthesizings

$$[\textsc{Not}\,[P_1]\cdots[P_k]]$$

and

$$[\textsc{Not}\,[[P_1]\cdots[P_k]]]$$

give the same result, and most of the negative compound propositions can be obtained in two different ways. The only case which is obtained in an unique way is when one only takes the negation of x_1. Therefore twice the number of negative compound propositions equals the total number of parenthesizings on a string of 11 elements

$$\textsc{Not}\, x_1x_2\cdots x_{10}$$

plus the total number of parenthesizings on a string of 10 elements

$$(\textsc{Not}\, x_1)x_2\cdots x_{10}.$$

This, indeed, provides the value $(s(10)+s(11))/2 = 310,954$.

Nowadays, the thinkers of Classic Antiquity do not enjoy the same authority and revered status as they had up to the 18th century. Armed with the machinery of enumerative combinatorics, we may look condescendingly at the fantastic technical achievement of Hipparchus (he had at his disposal just basic arithmetic and only rudimentary algebraic notation). But I find it highly significant that ancient Greek philosophers, as soon as they started to think about the logical structure of human thought, identified the problem of parsing and treated it mathematically.

13 Combinatorics as non-parametric mathematics

The belief that all simple (*having no continuous moduli*) *objects in the nature are controlled by the Coxeter groups is a kind of religion.*
V. Arnold [3]

I mentioned in Section 1 that the "kaleidoscope" definition of mirror systems can be weakened and made into an axiomatics of matroid theory. For that, we should not demand that our mirror system is closed and use instead an appropriate convexity property, making the whole theory even more geometrical in its spirit. This approach to matroid theory and its immediate and obvious generalization is developed in the book [9]. It is truly remarkable how naturally Coxeter Theory arises from the simplest, genuinely basic structures of combinatorics.

Many eloquent speeches were made, and many beautiful books written in explanation and praise of the incomprehensible unity of mathematics. In most cases, the unity was described as a cross-disciplinary interaction, with the same ideas being fruitful in seemingly different mathematical disciplines, and the technique of one discipline being applied to another. The *vertical* unity of mathematics, with many simple ideas and tricks working both at the most elementary and at rather sophisticated levels, is not so frequently discussed – although it appears to be highly relevant to the very essence of mathematical practice. I discuss matroids mostly because I wish to emphasize the "vertical" integrity of mathematics, linking the "local" and the "global" viewpoints.

Combinatorics studies structures on a finite set; many of the most interesting of these arise from elimination of continuous parameters in problems from other mathematical disciplines. Joseph Kung [35] characterized the corresponding areas of combinatorics as *non-parametric mathematics*.

For example, graphs appear in real life optimization problems as, say, sets of cities (vertices of the graph) connected by roads (edges of the graph) of a certain length. Combinatorics looks at the structure left after we ignore the lengths of the roads (which are continuous parameters in the original problem), as well as all topographical considerations, etc.. The combinatorial structure of the graph determines many important features of the original parametric problem. If we work on an optimal delivery problem it does matter, for example, whether our graph is connected or disconnected.

Matroid [41, 60] is a combinatorial concept which arises from the elimination of continuous parameters from one of the most fundamental notions of mathematics: that of linear dependence of vectors.

Indeed, let E be a finite set of vectors in a vector space $\mathbb{R}^n$. Vectors $\alpha_1, \dots, \alpha_k$ are linearly dependent if there exist real numbers $c_1, \dots, c_k$, not all of them zero, such that $c_1\alpha_1 + \cdots + c_k\alpha_k = 0$. In this context, the coefficients $c_1, \dots, c_k$ are continuous parameters; what properties of the set E remain after we decide never to mention them? The solution was suggested by Hassler Whitney [61] in 1936. He noticed that the set of linearly independent subsets of E has some very distinctive properties. In particular, if $\mathcal{B}$ is the set of *maximal* linearly independent subsets of E, then, by a well known result from linear algebra, it satisfies the following *Exchange Property*:

> *For all* $A, B \in \mathcal{B}$ *and* $a \in A \smallsetminus B$ *there exists* $b \in B \smallsetminus A$*, such that* $A \smallsetminus \{a\} \cup \{b\}$ *lies in* $\mathcal{B}$*.*

Whitney introduced the term *matroid* for a finite structure consisting of a set E with a distinguished collection $\mathcal{B}$ of subsets satisfying the Exchange Property. The origin of the word 'matroid' is in 'matrix': this is what is left of a matrix if we are interested only in the pattern of linear dependences of its column vectors. Since the Gaussian elimination procedure is about linear dependence, matroids naturally describe its combinatorial "skeleton".

Matroids arise in many areas of mathematics, including combinatorics itself. For example, when we take the set E of edges of a connected graph together with the collection $\mathcal{B}$ of its maximal trees, they happen to form a matroid. Moreover, the validity of the Exchange Property is almost self-evident and can be established by a simple combinatorial argument. However, there are deeper reasons why a matroid arises: it can be shown that the edges of a graph can be represented by vectors in such a way that linearly dependent sets of edges are exactly those containing closed cycles. The cohomological nature of the last observation should be apparent to everyone familiar with algebraic topology. This should not be surprising, since cohomology (with coefficients in $\mathbb{Z}$) is itself a classical example of elimination of continuous parameters and therefore belongs to non-parametric mathematics.

The work of three generations of mathematicians confirmed that matroids, indeed, capture the essence of linear dependence. Since linear dependence is a ubiquitous and really basic concept of mathematics, it is not surprising that the concept of matroid has proven to be one of the most pervasive and versatile in modern combinatorics.

It is an idea strongly promoted by Israel Gelfand that even such a simple object as a finite set should be endowed with some extra structure, and that the most fundamental structure on a finite set – even in the absence of any other structures – is provided by its symmetric group acting on it. The symmetric group already lurks between the lines of the Exchange Property in the form of transpositions (a, b) responsible for the exchange of elements.

The symmetric group Sym_n is the simplest example of a finite Coxeter group (or, equivalently, a finite reflection group). It can be interpreted geometrically as the group of symmetries of the regular $(n-1)$-dimensional simplex in $\mathbb{R}^n$ with the vertices

$$(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$$

resulting from permutation of vertices. In terminology of Coxeter groups and root systems, Sym_n has notation A_{n-1}.

We can replace the symmetric group with the reflection group C_n – the group of symmetries of another Platonic solid in $\mathbb{R}^n$, the n-cube $[-1, 1]^n$. Then we get a very natural generalisation of matroids, called *symplectic matroids*. We usually refer to matroids (in Whitney's classical sense) as *ordinary matroids*, to distinguish them from the more general symplectic matroids and from even more general Coxeter matroids.

Symplectic matroids are related to the geometry of vector spaces endowed with bilinear forms, although in a more intricate way than ordinary matroids to ordinary vector spaces: they come from isotropic subspaces of the space. The most interesting class (which we call Lagrangian) comes from Lagrangian subspaces, that is, isotropic subspaces of maximal possible dimension. Furthermore, Sym_n is naturally embedded in the group of symmetries of the n-cube, because we can make Sym_n permute the coordinate axes without changing their orientation; this action obviously preserves the n-cube $[-1, 1]^n$. Thus ordinary matroids can be also understood as symplectic (and, moreover, Lagrangian) matroids, the latter becoming the most natural generalisations of the former.

A general definition of Coxeter matroid can be best done in terms of collections of mirrors; unlike the case of finite reflection groups, we do not require from matroids that their systems of mirrors are *closed*, but still want the system of mirrors to have some nice geometric properties. Here is the definition:

> Let Δ be a convex polytope. For every edge of Δ, take the hyperplane that cuts the edge in its midpoint and is perpendicular to the edge and imagine this hyperplane being a semitransparent mirror (see Figure 11); denote by $\mathcal{M}$ the resulting collection of mirrors. Now mirrors from $\mathcal{M}$ multiply by reflecting in other mirrors, as in a kaleidoscope. We end up with the *closed* system of mirrors $\mathcal{M}^*$; if it contains only finitely many mirrors, we call Δ a *Coxeter matroid polytope*; this concept is cryptomorphically equivalent to that of a Coxeter matroid.

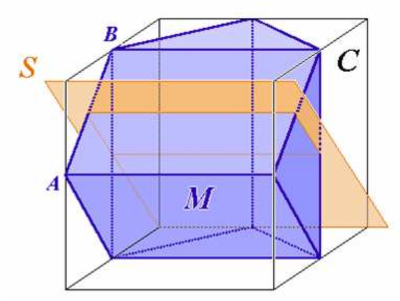


Figure 11 A Coxeter matroid polytope. After similarly drawing mirrors of symmetries for every edge, and reflecting mirrors in mirrors, we end up with the full system of mirrors of symmetries of the cube. Hence this matroid polytope corresponds to a symplectic matroid, that is, Coxeter matroid for the group C_3. (Drawing by Maria Borovik.)

Essentially, Coxeter matroids are n-dimensional kaleidoscopes which generate only finitely many mirror images.

However, this is only one of many possible definitions of Coxeter matroids. Being really basic mathematical structures, Coxeter matroids (and particularly the ordinary matroids, related to the symmetric group Sym_n viewed as the reflection group A_{n-1}) are notorious for their ability to cryptomorph and reappear in dozens of disguises which, at the first glance, appear to have nothing in common. This universality and ubiquity comes from the really elementary nature of Coxeter matroids: they are elementary particles, if you wish, of mathematical theories stripped of continuous parameters.

So far the story was strictly "local". It is interesting to follow Arnold [2] and retell it from the global point of view. I quote Arnold:

> Linear algebra is essentially the theory of the special roots system A_n. The basic facts of linear algebra (like the eigenvalues and the Jordan block theory) can be reformulated in terms of the roots, making the statements meaningful for other root systems. These new systems miraculously occur to be correct (while suitably modified). The theories of the root systems B_k, C_k, D_k (corresponding to the Euclidean and symplectic spaces geometry) are from this point of view *rather the sisters than the daughters* of the usual vector space geometry.

Arnold describes several ways of informal generalisation of huge tracts of mathematics, of which *symplectisation*, move from the root systems of type A_n to C_n, is a prominent case. He stresses that symplectisation is acting

> not on such small things, as points, functions, varieties, categories or functors, but on the whole of mathematics.

Also, according to Arnold, symplectisation (as well as complexification, quaternisation, etc.) is not confined to the finite-dimensional domain, and becomes even more interesting when applied to infinite-dimensional objects, see [2] for many colourful details.

However, it is Coxeter matroids for reflection groups other than A_n, especially symplectic matroids (for C_n), which provide the combinatorial underpinning for some of the "sister theories" of classical mathematics.

Acknowledgements

I thank my children Sergey and Maria, who corrected my English (further errors introduced by me are not their responsibility). I am grateful to my wife Anna, co-author and the harshest critic of our textbook on finite reflection groups [8]; neither [8], nor this paper would ever have appeared without her.

Gregory Cherlin, David Corfield, Chandler Davis, Eric Ellers, and Ray Hill carefully read and corrected the whole or parts of the paper.

My thanks are due to David Corfield for lively discussion of the philosophy of mathematics, to Satyan Devadoss for comments on the role of illustrations in mathematical texts, to John Stillwell for comments on the history of mathematics and to Paul Ernest, Reuben Hersh and Ann Kostant for encouraging comments.

An invitation to the conference *The Coxeter Legacy: Reflections and Projections* at the University of Toronto had a considerable influence on my work on this paper, and I am most grateful to its organizers.

While working on the paper, I used, on several occasions, the facilities of Mathematisches Forschungsinstitut Oberwolfach, the Fields Institute for Research in Mathematical Sciences and the Isaac Newton Institute for Mathematical Sciences.

Parts of the paper were written during my visit to University Paris VI in January 2004 on invitation from Michel LasVergnas, and I use this opportunity to tell Janette and Michel LasVergnas how enchanted I was by their hospitality.

During our conversation in Paris, Paul Moszkowski put forcefully the case for the development of the theory of Coxeter groups without reference to geometry and pointed me to his remarkable paper [39].

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