A Brief Introduction to Groups of Finite Morley Rank

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Introduction

Dimension, along with metric, topology, measure, etc., is one of the most basic mathematical concepts. It is also one of the oldest notions of Mathematics and can be traced back to Euclid. Besides the well-known use in Geometry and Topology, various versions of dimension have proved useful in Model Theory, in Combinatorics (the study of large highly transitive permutation groups Cherlin-Lachlan [21], Knight-Lachlan [34]) and Number Theory (in the proof of the Mordell-Lang conjecture for function fields Hrushovski [30]). Existence of a dimension theory for a certain class of mathematical objects imposes very strong restrictions on this class. For example, Hrushovski and Zil'ber have shown that the entire Algebraic Geometry can be reconstructed from an appropriate dimension theory [31, 32, 48].

The notion of Morley rank is a general notion of dimension as it emerged in Model Theory in the study of so-called $\aleph_1$-categorical structures (see Section 2 for a brief discussion of $\aleph_1$-categoricity). The best way to introduce Morley rank is to use an analogy with measure. We know that measure is a countably additive function

$$\mu : \Sigma \longrightarrow \mathbf{R}^{\geq 0} \cup \infty$$

defined on a certain set of sets Σ; the latter is supposed to be a σ-algebra, i.e. it is closed under complementation and countable unions.

For Morley rank we will use the same approach: *Morley rank* is a function

$$rk : \mathcal{U} \longrightarrow \mathbf{N}$$

defined for all non-emty sets in a set $\mathcal{U}$ of sets. We call $\mathcal{U}$ an *universe* and assume that $\mathcal{U}$ is closed with respect to certain set-theoretic operations. The

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axioms for an universe $\mathcal{U}$ and rank function rk are given in the next section. Our axioms are such that they are satisfied by dimension of constructible sets (i.e. Boolean combinations of Zariski closed sets) in Algebraic Geometry.

In this axiomatic framework we introduce groups of finite Morely rank. The theory of groups of finite Morley rank lies on the border between Group Theory and Model Theory. The basic notions of the theory are of model-theoretic origin, algebraic groups over algebraically closed fields constitute the main class of examples, and the most effective methods of exploration of groups of finite Morley rank are borrowed from Finite Group Theory. This synthetic nature of our theory makes any its non-formal exposition very difficult. Nevertheless in the present paper we attempt to meet the challenge and reach the frontiers of research after starting from axioms and basic definitions.

In Section 1 we introduce the axioms for Morley rank and discuss basic examples of groups of finite Morley rank, then in Section 2 briefly review model-theoretic connections of our theory. Section 3 contains the most important known results about special classes of groups of finite Morley rank. Section 4 is devoted to the the main conjecture in the theory. It is due to Gregory Cherlin and Boris Zil'ber and says that every infinite simple group of finite Morley rank is an algebraic group over an algebraically closed field. The last three Sections 5 – 7 discuss results in the area obtained by the Summer of 1994. A more up-to-date treatment of the subject can be found in the subsequent and simultaneously distributed text, *Tame groups of odd and even type.*

Almost all the results and concepts mentioned in the present paper may be found in the book by Borovik and Nesin [17] and in the paper [15]. We use many ideas and concepts of Finite Group Theory; the reader may wish to consult Gorenstein [27] for introduction to the classification theory of finite simple groups. The analogy with, as well as some technical results from, Algebraic Group Theory are also very important for the present paper; the most convenient reference is Humphreys [33] (this book also contains a brief introduction to Algebraic Geometry). We mention in Section 2 ω-stable groups; the reader who wish learn more about them can consult Poizat [42]. We recommend Chang and Keisler [20] and Hodges [29] for references concerning Model Theory.

1 Axioms of Morley Rank

Universe. A *universe* is a collection $\mathcal{U}$ of sets that satisfies certain properties that we will soon list. We will refer to the elements of $\mathcal{U}$ as the *definable* sets. A function will be called *definable* if and only if its graph is. We require the following axioms from a universe:

- *Closure under Boolean operations.* If A and B are definable sets, then the sets $A \cap B$, $A \cup B$ and $A \setminus B$ are also definable.

- *Closure under products.* If A and B are definable sets, then their Cartesian product $A \times B$ and the canonical projections

$$\pi_1 : A \times B \to A, \quad \pi_2 : A \times B \to B$$

  are also definable. If $A = B$, then the diagonal

$$\Delta = \{(a, a) \mid a \in A\} \subset A \times A$$

  is also definable. We assume also that if C is a definable subset in $A \times B$, then the images $\pi_1(C), \pi_2(C)$ of C under canonical projections are definable.

- *Finite subsets.* If A is definable and $a \in A$, then the singleton set $\{a\}$ is definable.

- *Factorization.* If $E(x, y)$ is a definable equivalence relation on a definable set A (thus in particular E is a definable subset of A^2), then the quotient A/E is encoded in the universe as follows: there is a set $\bar{A} \in \mathcal{U}$ and a surjective definable function $f : A \longrightarrow \bar{A}$ such that for $x, y \in A$ we have: $f(x) = f(y)$ iff $E(x, y)$ holds.

It easily follows from these axioms that the union and the intersection of finitely many definable sets are also definable, that the emptyset $\emptyset$ and all the finite and cofinite subsets of a definable set are definable. Also the image and the preimage of a definable function are definable sets.

Given a set of sets $\{A, B, C, \ldots\}$, there is the minimal universe $\mathcal{U}$ containing these sets; we shall say that $\mathcal{U}$ is *generated* by the sets $A, B, C, \ldots$. If we have an algebraic structure, say, a group G, we can generate by G the universe $\mathcal{U}(G)$ in the following way: take the graph $M \subset G \times G \times G$ of the multiplication function $G \times G \longrightarrow G$, $(g, h) \mapsto gh$ and the graph $I \subset G \times G$ of the inversion function $G \longrightarrow G$, $g \mapsto g^{-1}$, and set $\mathcal{U}(G)$ to be equal to the universe generated by the underlying set of G, M and I.

We also say that an algebraic structure is *definable* or (what is the same) *interpretable* in the universe $\mathcal{U}$, if its underlying set, operations and predicates belong to $\mathcal{U}$. If a structure $\mathcal{N}$ is interpretable in $\mathcal{U}(\mathcal{M})$, then we say that $\mathcal{N}$ is *interpretable* in $\mathcal{M}$ and $\mathcal{M}$ *interprets* $\mathcal{N}$. For example (and the reader may wish to check this, it is an easy exercise), if $N \lhd G$ is a normal definable subgroup of a group G, then the factor group G/N is interpretable in G.

Rank. Let $\mathcal{U}$ be a universe. A function

$$rk : \mathcal{U} \setminus \{\emptyset\} \longrightarrow \mathbf{N}$$

is called a *rank* if the following axioms are satisfied for all A, $B \in \mathcal{U}$.

- *Monotonicity of rank.* $rk(A) \geq n + 1$ if and only if there are infinitely many pairwise disjoint, non-empty, definable subsets of A each of rank at least n.
- *Definability of rank.* If f is a definable function from A into B, then, for each integer n, the set $\{b \in B \mid rk(f^{-1}(b)) = n\}$ is definable.
- *Additivity of rank.* If f is a definable function from A onto B and if for all $b \in B$, $rk(f^{-1}(b)) = n$ then $rk(A) = rk(B) + n$.
- *Boundness of finite preimages.* For any definable function f from A into B there is an integer m such that for any b in B the preimage $f^{-1}(b)$ is infinite whenever it contains at least m elements.

We say that a universe $\mathcal{U}$ is *ranked* if there is a rank function with the above properties.

If $\mathcal{M}$ is an algebraic structure, we say that $\mathcal{M}$ is a *ranked structure* if $\mathcal{U}(\mathcal{M})$ is a ranked universe. In this case, the rank of the definable set M is called the *rank* of the structure $\mathcal{M}$.

Groups of finite Morley rank. Now we can give our main definition:

> *A group G is said to be of finite Morley rank if it is ranked.*

We also say that a field K has finite Morley rank if the universe $\mathcal{U}(K)$ (generated, according to our definition, by the underlying set of K and by the graphs of addition, multiplication and inversion on K) is ranked. We will return to discussion of the universe $\mathcal{U}(K)$ later.

Examples of groups of finite Morley rank. We give five types of examples of groups of finite Morley rank.

- *Finite groups.* They obviously have Morley rank 0.
- *Algebraic groups.* An algebraic group G over an algebraically closed field K is a group of finite Morley rank (and the rank of G itself coincides with the dimension of G over K).
- *Abelian groups of bounded exponent.* It can be proven that they have finite Morley rank.
- *Divisible abelian groups.* A divisible abelian group A is of finite Morley rank if and only if for every prime number p it contains only finitely many quasicyclic direct factors $\mathbf{Z}(p^\infty)$. In particular, torsion-free divisible abelian groups are of finite Morley rank, as well as the quasicyclic groups $\mathbf{Z}(p^\infty)$. (Recall that the *quasicyclic p-group* $\mathbf{Z}(p^\infty)$ is isomorphic to the group of all complex roots $\sqrt[p^n]{1}$, $n = 1, 2, \ldots$, of 1.)

- *Direct products.* It can be shown that a direct product of two groups of finite Morley rank is a group of finite Morley rank.

And here are some examples of groups which *are not* groups of finite Morley rank:

- The infinite cyclic group $\mathbf{Z}$.
- Free groups F_n.
- Real orthogonal groups $\mathrm{SO}_n(\mathbf{R})$, $n \geq 3$, and, more generally, simple compact Lie groups.

Fields of finite Morley rank. By a remarkable result due to Angus Macintyre [36], an infinite field has finite Morley rank if and only if it is algebraically closed.

Definability in Model Theory. The notion of Morley rank originates in Model Theory and we will briefly discuss model-theoretical connections of our theory.

Model theory of groups deals mostly with structural properties of groups, their subgroups and subsets which can be expressed by formulae of the first-order logic.

Let M be an algebraic structure (say, a group or a field). A subset $X \subseteq M^n$ of a finite direct power M^n of M is called *definable*, if there is a formula $\phi(x_1, \ldots, x_n, a_1, \ldots, a_m)$ in the language of M (in the case of groups, for example, this means that the only functional symbols involved in ϕ are $\cdot$ and $^{-1}$) with free variables $x_1, \ldots, x_n$, and parameters $a_1, \ldots, a_m \in M$ such that

$$X = \{(b_1, \ldots, b_n) \in M^n \mid \phi(b_1, \ldots, b_n, a_1, \ldots, a_m)\}.$$

For example, if G is a group and $a \in G$, then the centralizer $C_G(a)$ and the conjugacy class a^G are definable in G (using the parameter a):

$$\begin{aligned} C_G(a) &= \{g \in G \mid ga = ag\}, \\ a^G &= \{g \in G \mid \exists y (y^{-1} a y = g)\}, \end{aligned}$$

the corresponding formulae here are $xa = ax$ and $\exists y(y^{-1}ay = x)$. The center $Z(G)$ of the group G is definable in G *without parameters*:

$$Z(G) = \{g \in G \mid \forall y (yg = gy)\}.$$

If two definable subsets $X, Y \subseteq G$ are given by the formulae $\phi(x)$, $\psi(x)$, correspondingly, then the formulae

$$\phi(x) \wedge \psi(x), \phi(x) \vee \psi(x),\ \phi(x) \wedge \neg\psi(x),\ \exists u \exists v (x = uv \wedge \phi(u) \wedge \psi(v))$$

defines the sets $X \cap Y$, $X \cup Y$, $X \setminus Y$, and $X \cdot Y$. (An exercise for the reader: if $X \subseteq G$ is defined by $\phi(x)$, write formulae for the centralizer $C_G(X)$ and the normalizer $N_G(X)$, thus proving that these two subgroups are also definable.)

Given an algebraic structure M, one can show that every definable in M subset of M^n belongs to $\mathcal{U}(M)$, i.e. is definable in $\mathcal{U}(M)$ in the sense of our axiomatic definition. Moreover, if we define *interpretable* in M sets as the quotients of definable in M sets modulo definable relations, then one can identify the universe $\mathcal{U}(M)$ with the set of all interpretable (in M) sets.

This can be illustrated by the following example. Let G be a group. As we already know, the universe $\mathcal{U}(G)$ is generated by the underlying set of G (which we denote by the same symbol G) and by the graphs $M \subset G \times G \times G$ and $I \subset G \times G$ of the multiplication function $G \times G \longrightarrow G$ and the inversion $G \longrightarrow G$, so

$$M = \{(x, y, xy) \mid x, y \in G\}.$$

The set I of all involutions (i.e. elements of order 2) in G is definable by a very simple formula $x^2 = 1 \wedge \neg x = 1$. But it is also can be built in $\mathcal{U}(G)$ as follows. Take the diagonal Δ of $G \times G$ and form the subset $X = M \cap (\Delta \times G)$ of $G \times G \times G$; let π_1 and π_3 be the canonical projections of $G \times G \times G$ on the first and the third component, then $I = \pi_1(X \cap \pi_3^{-1}(1)) \setminus \{1\}$.

A special case of this construction is the *universe of constructible sets* $\mathcal{U}(K)$ over an algebraically closed field K. Indeed, Alfred Tarski proved in [43] that sets, definable in an algebraically closed field K by formulae of the first-order language are *constructible*, i.e. they are Boolean combinations of Zariski closed subsets of K^n, $n = 1, 2, \ldots$ [1], the validity of all, but the last, axioms for them is obvious. Bruno Poizat proved in [41] that the quotients of a constructible set with respect to a constructible equivalence relation (as in the last axiom) yield constructible sets.

For example, if $K = \mathbf{C}$ is the field of complex numbers, then the set $U \subset \mathbf{C}^3$ of all coefficients u, v, w of the quadratic equations $ux^2 + vx + w = 0$ with non-repeated roots is definable by the formula

$$\exists x_1 \exists x_2 (ux_1^2 + vx_1 + wx_1 \wedge ux_2^2 + vx_2 + wx_2 \wedge \neg x_1 = x_2);$$

in the same time U is the constructible set

$$U = \mathbf{C}^3 \setminus \{u, v, w \mid v^2 - 4uw = 0\}.$$

Our next example illustrates the quotients of constructible sets. The projective line $\mathbf{P}^1(\mathbf{C})$ is the quotient of the definable (and constructible) set $\mathbf{C}^2 \setminus \{(0, 0)\}$ modulo the definable equivalence relation

$$(x, y) \sim (\lambda x, \lambda y),\ \lambda \neq 0,$$

but in the same time can be identified with the constructible set $\mathbf{C} \cup \{\infty\}$.

[1] A *Zariski closed* set in K^n is the solution set, over the field K, of a system of polynomial equations in n variables with coefficients in K.

2 Link to Model Theory: ω-stable groups of finite Morley rank

From the model-theoretic point of view groups of finite Morley rank are a special case of *ω-stable* groups. The concept of ω-stability arose in Model Theory in the late 1960s. It is a very subtle notion and we are not in a position to discuss it here. However a modicum of model-theoretical terminology is unavoidable in any discussion of the subject.

In an ω-stable structure M every definable set can be assigned an ordinal which is called its *Morley rank*, and a group G has finite Morley rank in the sense of our axiomatic definition if and only if G is ω-stable and the ranks of all definable in G sets are finite. There are also interesting examples of ω-stable structures of *infinite Morley rank* occurring in algebra, notably the *differentially closed fields* of characteristic zero.

One of the main problems in the theory of ω-stable groups is classification of simple groups of finite Morley rank. About fifteen years ago Gregory Cherlin and Boris Zil'ber conjectured a possible solution to this problem.

The Cherlin-Zil'ber Conjecture. *Simple infinite groups of finite Morley rank are algebraic groups over algebraically closed fields.*

This is certainly the central and the most important problem of the theory of ω-stable groups. The present paper is devoted to discussion of possible approaches to its resolution.

$\aleph_1$-categorical groups If we restrict our attention to *simple* groups of finite Morley rank, then this class of groups can be given a relatively simple characterization in terms of another model-theoretical concept, categoricity.

Let $\mathrm{Th}(G)$ denote the set of all sentences of the first-order language which are true in G. Two groups G and H are called *elementarily equivalent* if $\mathrm{Th}(G) = \mathrm{Th}(H)$. For example, since the property of *commutativity* is expressed by the formula

$$\forall x \forall y (xy = yx),$$

divisibility by the infinite sequence of formulae

$$\forall x \exists y (y^2 = x), \forall x \exists y (y^3 = x), \ldots, \forall x \exists y (y^n = x), \ldots,$$

and *torsion-freeness* by the sequence of formulae

$$\forall x (x^2 = 1 \to x = 1), \forall x (x^3 = 1 \to x = 1), \ldots, \forall x (x^n = 1 \to x = 1), \ldots,$$

it follows that any group elementarily equivalent to a torsion-free divisible abelian group is itself torsion-free, divisible, and abelian; and conversely, one can show that any two nontrivial torsion-free divisible abelian groups are indeed elementarily equivalent.

It is known from Model Theory that for every infinite group G there is a group $\tilde{G}$ elementarily equivalent to G and of cardinality $\aleph_1$. A group G is called $\aleph_1$-*categorical*, if $\tilde{G}$ is unique (up to isomorphism).

Since torsion-free divisible abelian groups of cardinality $\aleph_1$ are vector spaces over $\mathbf{Q}$ of dimension $\aleph_1$ and so are isomorphic, nontrivial torsion-free divisible abelian groups are $\aleph_1$-categorical. In particular, the additive group $\mathbf{Q}^+$ of rational numbers is $\aleph_1$-categorical.

Algebraically closed fields are also $\aleph_1$-categorical: in each characteristic there is exactly one algebraically closed field of cardinality $\aleph_1$, up to isomorphism. It can be deduced from this that simple algebraic groups over algebraically closed fields are $\aleph_1$-categorical.

It follows from deep model-theoretic results by Baldwin and Zil'ber that classes of simple groups of finite Morley rank and simple $\aleph_1$-categorical groups coincide. So the Cherlin-Zil'ber conjecture can be understood as a conjecture about simple $\aleph_1$-categorical groups (indeed Zil'ber originally stated his conjecture in this form). Unfortunately the class of $\aleph_1$-categorical groups is not closed under passage to definable subgroups. On the other hand a definable subgroup of a group of finite Morley rank is again a group of finite Morley rank. For this reason it is much more convenient to work in the class of groups of finite Morley rank than in the narrower class of $\aleph_1$-categorical groups.

3 Basic properties of groups of finite Morley rank

When working with ranked groups it is useful to think of them as algebraic groups from which the structure of an algebraic variety has been deleted, while retaining a notion of dimension for the constructible sets. Many group-theoretical constructions in a ranked group yield definable sets: intersections, products, as well as the centralizers and normalizers of definable subgroups are definable. Also definable are the centralizers of elements. This analogy will suffice for an understanding of the subsequent text without turning to the book by Borovik and Nesin [17], which contains proofs of all the necessary auxiliary results. The following facts illustrate the analogy with algebraic groups.

From now on G stands for a group of finite Morley rank.

General structural properties. Groups of finite Morley rank satisfy many group-theoretical minimality properties, and most of them follow from the following general result.

Theorem 1 (Macintyre [35]) *A group G of finite Morley rank satisfies the descending chain condition for chains of definable subgroups.*[2]

In particular, we have

Corollary 2 *A group G of finite Morley rank contains a unique minimal definable subgroup of finite index, which is called* the connected component of G *and is denoted by G°.*

A group G is called *connected*, if $G = G^\circ$.

Corollary 3 (Baldwin and Saxl [9]) *The centralizer $C_G(X)$ of any subset X in a group of finite Morley rank is a definable subgroup. Moreover, G satisfies the ascending and descending chain conditions for centralizers.*

The following fact is a consequence of a deep result by Boris Zil'ber, so called Zil'ber's Indecomposability Theorem [47]. It yields the definability of a wide range of subgroups.

Theorem 4 (Zil'ber [47]) *If $\{H_i \mid i \in I\}$ is a family of connected definable subgroups in a group G of finite Morley rank, then the subgroup $\langle H_i \mid i \in I\rangle$ generated by them is definable and connected.*

Special classes of groups. We list here some known results on abelian, nilpotent, solvable and locally finite groups of finite Morley rank.

Abelian groups. In the abelian case we have a complete understanding of the situation.

Theorem 5 (Macintyre [35]) *Let G be an abelian group of finite Morley rank. Then $G = D \oplus B$ where D is a divisible group and B is a subgroup of bounded exponent.*

Nilpotent groups. A great deal is also known about nilpotent groups of finite Morley rank.

Theorem 6 (Nesin [38]) *Let G be a nilpotent group of finite Morley rank. Then G is a central product $D * C$ where D is definable, connected, characteristic in G and divisible, C is definable and of bounded exponent.*

Solvable groups. Here we also have useful structural information. The following is only one of many results, it generalises to groups of finite Morley rank a well-known property of solvable algebraic groups.

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[2] This means that any descending chain of definable subgroups

$$H_1 \geq H_2 \geq \dots$$

stabilises after finitely many steps.

Theorem 7 (Zil'ber [46], Nesin [37]) *Let G be a solvable connected group of finite Morley rank. Then the commutator subgroup G' is nilpotent.*

For solvable groups of finite Morely rank we also have analogues of the theorems on Hall and Sylow subgroups from Finite Group Theory (Altınel, Cherlin, Corredor and Nesin [4], Borovik and Nesin [16, 18]). Unfortunately due lack of space we are not in a position to discuss these results in the present paper.

Locally finite groups. The complete description of simple infinite locally finite groups of finite Morley rank was obtained by Simon Thomas with the use of the classification of finite simple groups. This is one of the most important sources of inspiration for our theory.

Theorem 8 (Thomas [44]) *A simple infinite locally finite group G of finite Morley rank is isomorphic to a simple algebraic group over the algebraic closure $\overline{\mathbf{F}}_p$ of a finite prime field $\mathbf{F}_p$.*

4 Cherlin-Zil'ber Conjecture

We would like to discuss in this section a possible approach to the Cherlin-Zil'ber Conjecture. In what follows G is a connected group of finite Morley rank.

In a search of a field. In any attempt to identify a simple group G of finite Morley rank with an algebraic group over an algebraically closed field we need, first of all, to construct this field. It can be done in *solvable non-nilpotent subgroups* of G (if G contains any) by means of the following result, due to Boris Zil'ber.

Theorem 9 (Zil'ber [47]) *A solvable connected non-nilpotent group G interprets an algebraically closed field.*

Zil'ber's theorem is not applicable to connected solvable subgroups of G if all of them are *nilpotent*. This cannot happen in simple algebraic groups G where Borel subgroups (i.e. maximal connected solvable closed subgroups) are known to be non-nilpotent [33, §21.4].

The notion of bad groups formalises a situation which is furthest away from the realm of algebraic groups. By definition, a *bad group* is a non-solvable group of finite Morley rank whose proper, definable and connected subgroups are nilpotent.

The following simple result shows the place of bad groups in the theory:

Theorem 10 (Borovik and Nesin [17, Proposition 13.2]) *Let G be a connected non-nilpotent group of finite Morley rank. Then either an algebraically closed field is interpretable in G or G interprets a simple bad group.*

It is not known whether or not bad groups exist. Their possible existence is one of main obstacles to the classification of simple groups of finite Morley rank.

Bad Group Problem (G. Cherlin) *Are there any bad groups?*

Bad fields. By definition, a *bad field* is a ranked structure of the form

$$\langle K,\, +,\, \cdot\,,\, 0,\, A\rangle$$

where $\langle K,\, +,\, \cdot\, \rangle$ is an algebraically closed field and A is a (predicate for a) proper infinite multiplicative subgroup of K^*.

Bad Field Problem. (B. Poizat) *Are there bad fields?*

If a bad field $\langle K,\, +,\, \cdot\,,\, 0,\, A\rangle$ exists, then the group S of matrices over K of the form

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a \in A, b \in K,$$

has finite Morley rank and solvable, but not algebraic. One of the worst things that may happen is that S does not necessary contain an involution (i.e. an element of order 2). But if S is algebraic (i.e. $A = K^*$), S does contain an involution.[3]

Tame groups. Let us call a group G of finite Morley rank *tame*, if it does not interpret a bad group or a bad field.

Tame Conjecture (1). *Every tame infinite simple group of finite Morley rank is an algebraic group over an algebraically closed field.*

Obviously the negative answers to the Bad Group and Bad Field Problems, and the confirmation of the Tame Conjecture would provide the complete proof of the Cherlin-Zil'ber Conjecture.

We will concentrate in this paper on discussion of the recent progress in the study of tame groups.

The starting point of the theory of tame groups is the following analogue of the Feit-Thompson theorem on solvability of groups of odd order: a connected tame group without involutions is nilpotent!

Theorem 11 (Borovik and Nesin [17, Theorem B.1]) *Any tame connected definable group G without involutions is nilpotent.*

This result shows that the Bad Group and Bad Field Problems play in the theory of groups of finite Morley rank the same role as Burnside's question on

[3] These statements can be left to the reader as exercises.

solvability of finite groups of odd order in the theory of of finite groups; the question was answered, more than 50 years later, by Feit and Thompson in their famous "Odd order paper" [25].

The idea of a *minimal counterexample* borrowed from the theory of finite groups yields remarkable simplifications in all considerations concerned with the Tame Conjecture. Indeed, we can restrict ourselves to dealing with a group G of minimal Morley rank subject to being a counterexample to the Tame Conjecture. Then we can assume without loss that every proper, simple, definable and connected section[4] S of G is a simple algebraic group over some algebraically closed field. It will be convenient to use the following two technical definitions describing this situation.

A group G of finite Morley rank is called a *K-group*, if every infinite, simple, definable and connected section of G is an algebraic group over an algebraically closed field.

We shall also call a group of finite Morley rank G a *K*-group*, if every *proper* definable subgroup of G is a K-group.

So in order to confirm the Tame Conjecture it suffices to prove its version for K*-groups.

Tame Conjecture (2). *Every infinite simple tame K*-group is a simple algebraic group over an algebraically closed field.*

5 Sylow Theory and Odd/Even Dichotomy

2-Sylow Theorem. A *Sylow p-subgroup* of a group G is a maximal p-subgroup in G. Obviously every p-subgroup of G lies in some Sylow p-subgroup.

Sylow 2-subgroups in groups of finite Morley rank are not necessary definable. An easy example is provided by the group $\mathrm{SL}_2(\mathbf{C})$: here any Sylow 2-subgroup is conjugate to the group S of monomial matrices of the form

$$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & 0 \end{pmatrix} \right\},$$

where $\lambda^{2^n} = 1$ for some $n \in \mathbf{N}$.

S contains a subgroup T of index 2,

$$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\};$$

T is a divisible abelian group and has a property that every proper subgroup of T is a finite cyclic 2-group. Obviously $T \simeq \mathbf{Z}(2^\infty)$ is the quasicyclic 2-group.

[4] A *section* of a group G is the factor group H/N for some subgroups $N \lhd H$ of G; the section is *definable* if the both H and N are definable in G.

Moreover, an analogous fact is valid for any algebraic group G over an algebraically closed field of characteristic not 2: a Sylow 2-subgroup P in G is not definable and contains a Prüfer 2-subgroup T of finite index. (A *Prüfer 2-group* is a product of a finite number of copies of the quasicyclic group $\mathbf{Z}(2^\infty)$.) Moreover, it can be shown that P is not nilpotent though it is solvable and abelian-by-finite. The exponent of P is infinite and all elementary abelian subgroups of P are finite.

On the contrary, in the case of characteristic 2 Sylow 2-subgroups in semisimple algebraic groups are precisely maximal unipotent subgroups. Therefore they are definable, nilpotent, connected, have bounded exponents and contain infinite elementary abelian 2-subgroups.

Theorem 12 (Borovik and Poizat [19]) *All Sylow 2-subgroups in a group G of finite Morley rank are conjugate. If S is one of them, then S has a subgroup S° of finite index with the following property: $S^\circ = B * D$ is a central product of a definable connected nilpotent subgroup B of bounded exponent and of a divisible abelian group D. Subgroups B and D are uniquely determined in S. In particular, S is nilpotent-by-finite.*

We shall call B the *bounded* or *unipotent part* of S and D the *maximal 2-torus* of S. More generally, any divisible abelian 2-subgroup of G is called *2-torus*. 2-tori in groups of finite Morley rank are Prüfer groups.

Theorem 13 (Borovik and Nesin [17, Theorem B.3]) *Let G be a simple tame group. Then the Sylow 2-subgroups in G are infinite.*

Groups of odd and even type. First of all, we have to find some way to separate properties of 2-Sylow subgroups arising from algebraic groups of characteristic 2 and not 2. Tuna Altınel has recently came very close to completion of a proof for the following theorem.

Theorem 14 [2]) *Let S be a Sylow 2-subgroup of a simple tame K^*-group G and $S^\circ = T * B$, where T is the maximal 2-torus and B is the bounded part of S. Then either $T = 1$ or $B = 1$.*

In the two cases of Theorem 14, we say that G is of *even type* if $T = 1$ and of *odd type* if $B = 1$. So we arrive at the most important bifurcation point of our theory:

Odd/Even Dichotomy. *Every simple tame group of finite Morley rank is either of odd or even type.*

An analogy with Finite Group Theory suggests that these two classes of groups should be studied by completely different methods. In the present paper we restrict our attention (due to lack of space) to the better understood class of groups of odd type.

2-generated core. If U is a finite abelian 2-group, $m(U)$ stands for the minimal number of generators for U. If G is an arbitrary group, its *2-rank* $m(G)$ is defined as the maximum of $m(U)$ for all finite abelian 2-subgroups $U \leq G$. It can be shown that for a group G of finite Morley rank and odd type $m(G)$ is finite.

We also introduce the notion of the *normal 2-rank* $n(G)$ of a group G of odd type (or, what is the same, of its Sylow 2-subgroup): if S is any Sylow 2-subgroup of G, $n(G) = n(S)$ is set to be equal to the maximum of 2-ranks $m(E)$ of *normal* abelian subgroups $E \trianglelefteq S$.

For example, if P is a Sylow 2-subgroup in $G = \mathrm{PSL}_2(\mathbf{C})$, then it can be easily shown that $m(P) = 2$ and $n(P) = 1$. (P is the factor group of a Sylow 2-subgroup S in the group $\mathrm{SL}_2(\mathbf{C})$, discussed in the beginning of this section, modulo the subgroup $\langle diag(-1,-1)\rangle$ of order 2. It is worth mentioning that for S the 2-rank and normal 2-rank coincide: $m(S) = n(S) = 1$.)

Let S be a Sylow 2-subgroup in a group G of finite Morley rank. We define the *2-generated core* $\Gamma_{S,2}(G)$ as the definable closure of the group generated by all normalizers $N_G(U)$ for all subgroups $U \leq S$ with $m(U) \geq 2$. (Here the *definable closure* of a set $X \subset G$ is the intersection of all definable subgroups containing X.) Finite simple groups with a proper 2-generated core $\Gamma_{S,2}(G) < G$ are known by Aschbacher [6]. As the reader will see from the sequel, in the finite Morley rank context an analogous result plays a even more important role. Fortunately, we do not expect existence of proper 2-generated cores in simple infinite groups of finite Morley rank.

6 Subgroups of a Minimal Counterexample

Components. A group G will be called *quasi-simple*, if $G = G'$ and $G/Z(G)$ is non-abelian simple. Note that the only proper normal subgroups of a quasi-simple group are the central ones. Notice that quasi-simple *algebraic groups* are traditionally called *simple algebraic groups.*

Theorem 15 (Belegradek [10]) *In a group G of finite Morley rank, every quasi-simple subnormal subgroup is definable and there are only finitely many of them. We shall call them* components *of G. The product $L(G)$ of components of G is a definable normal subgroup in G and every component of G is a normal subgroup of $L(G)$.*

The subgroup $L(G)$ is called the *layer* of G. We also denote $E(G) = L(G)^\circ$. We list for convenience some further properties of the layer:

Theorem 16 [17, Lemma 7.10]

(i) *If $H \lhd G$ is a definable subgroup, then $L(H) \lhd G$ and definable.*

(ii) *If G is connected, then any quasi-simple subnormal subgroup of G is normal in G and connected.*

(iii) *$E(G) = L(G^\circ)$ and all the components of $E(G)$ are connected.*

Generalised Fitting Subgroup. Let G be any group. Let $F(G)$ be the subgroup generated by all the normal nilpotent subgroups. Clearly $F(G)$ is a characteristic subgroups. It is called the *Fitting subgroup* of G.

Theorem 17 (Belegradek [10] and Nesin [39]) *Let G be a group of finite Morley rank. Then $F(G)$ is a definable nilpotent subgroup.*

We will now define the *generalised Fitting subgroup* $F^*(G)$ of a group G of finite Morley rank as

$$F^*(G) = F(G)L(G).$$

Clearly $F^*(G)$ is a definable characteristic subgroup of G. This notion exists already in Finite Group Theory and was introduced by Bender [11].

Structural properties of K-groups. Let H be a group of finite Morley rank. We shall denote by $O(H)$ the maximal normal definable connected subgroup of H *without involutions*. It exists because it may be shown that the product of two normal definable subgroups without involutions does not again possesses involutions.

Theorem 18 *If H is a tame K-group, then the subgroup $O(H)$ is nilpotent. If, furthermore, H is of odd type and $\overline{H} = H/O(H)$, then $\overline{H}^\circ = F^*(\overline{H})$ and $F(\overline{H})$ is an abelian group. Moreover, simple components of $F(\overline{H})$ are simple algebraic groups over algebraically closed fields of characteristic not 2.*

The proof of the above result involves the following remark by Tuna Altınel and gregory Cherlin on central extensions of simple algebraic groups.

Theorem 19 (Altınel and Cherlin [3]) *Let G be a quasisimple group of finite Morley rank with $G/Z(G)$ isomorphic to a simple algebraic group over an algebraically closed field. Assume that G does not interpret a bad field. Then G is algebraic.*

7 Groups of Odd Type

In this section we start a systematic study of groups of odd type. First of all we restate the Tame Conjecture for the class of groups of odd type.

Tame Conjecture (3) *All simple tame K^*-groups of odd type are algebraic groups over algebraically closed fields of characteristic not 2.*

It has been already mentioned that 2-rank $m(G)$ of a group G of odd type $m(G)$ is finite. Therefore G has also finite normal rank $n(G)$.

Centralizers of involutions in groups of odd type. If t is an involution we abbreviate $C_t = C_G(t)$.

Notice, first of all, that in a simple group of finite Morley rank the centralizer of any involution is infinite; this follows from the following simple result.

Theorem 20 (Borovik [13]). *If the centralizer C_t of an involution t in a group of finite Morley rank G is finite, then G° is abelian, $t \in G \setminus G^\circ$ and t inverts every element in G°, i.e. $g^t = g^{-1}$ for all $g \in G^\circ$.*

The first step in classification of groups of odd type is to show that the structure of centralizers of involutions resembles the structure found in one of the simple algebraic groups of characteristic not 2. We shall say that G satisfies the *B-conjecture*, if $C_t^\circ = F(C_t)^\circ E(C_t)$ for any involution $t \in G$. This notion generalises to the context of finite Morley rank the B-conjecture by J. Thompson from Finite Group Theory. It easily follows from the structural properties of algebraic groups that simple algebraic groups over algebraically closed fields of characteristic not 2 satisfy the B-conjecture.

When we ask the centralizers of involutions in simple groups of odd type to resemble the centralizers of involutions in simple algebraic groups of characteristic not 2, we mean, first of all, the validity of the B-conjecture.

Theorem 21 *Let G be a simple tame K*-group of finite Morley rank and odd type. If $t \in G$ is an involution, then $O(C_t)$ is a nilpotent group and $\overline{C_t^\circ} = C_t^\circ/O(C_t)$ is a central product of finitely many simple algebraic groups over algebraically closed fields of characteristic not 2 and an abelian divisible group. In particular, $\overline{C_t^\circ} = F(\overline{C_t^\circ})E(\overline{C_t^\circ})$.*

We see that one of the ways to prove the B-conjecture for groups of finite Morley rank and odd type is to show that in a tame simple K*-group G, $O(C_t) = 1$ for all involutions $t \in G$.

When G has normal 2-rank $n(G) \geq 3$, we can develop an approach to this problem based on the well-known idea of the signalizer functor, which will be discussed later. But if the normal 2-rank of G is 1 or 2, we do not have in our possession this powerful method. We would like to emphasise the challenge posed by groups of small normal rank. Their classification will be a substitute in our theory for the theorem by Gorenstein and Harada [28] on finite groups of the so-called sectional 2-rank ≤ 4. The expected list of simple tame K*-groups G with $n(G) \leq 2$ consists of the groups $\mathrm{PSL}_2(K)$, $\mathrm{PSL}_3(K)$, $\mathrm{PSp}_4(K)$, $\mathrm{G}_2(K)$ over some algebraically closed field K of characteristic not 2.

Signaliser functor. A possible approach to proving that $O(C_t) = 1$ for involutions $t \in G$ is well-known to the finite group theorists and based on the following observation.

Theorem 22 [17, Theorem B.29]. *For any two commuting involutions t, s in a tame K^*-group G we have*

$$O(C_t) \cap C_s = O(C_s) \cap C_t.$$

In this situation, the finite group theorists say that $\theta(t) = O(C_t)$ is a *signalizer functor*. More precisely, for any involution $s \in G$, let $\theta(s) \leq O(C_s)$ be some connected definable normal subgroup of C_s. We say that θ is a signalizer functor, if for any commuting involutions $t, s \in G$

$$\theta(t) \cap C_s = \theta(s) \cap C_t.$$

The signalizer functor θ is *complete*, if for any elementary abelian subgroup $E \leq G$ of order ≥ 8 the subgroup

$$\theta(E) = \langle \theta(t),\, t \in E^* \rangle$$

is a connected subgroup without involutions and

$$C_{\theta(E)}(s) = \theta(s)$$

for any $s \in E^*$.

And, finally, a signalizer functor θ is *non-trivial*, if $\theta(s) \neq 1$ for some involution $s \in G$, and *nilpotent*, if all the subgroups $\theta(t)$ are nilpotent.

The following theorem is a generalisation of results of David Goldschmidt [26] and Helmut Bender [12] on signalizers in finite groups.

Theorem 23 (Borovik [14]) *Any nilpotent signalizer functor θ on a group G of finite Morley rank is complete.*

Now we can state our main result about signalizer 2-functors.

Theorem 24 (Borovik [15]) *In the notation above, if a group G of finite Morley rank and odd type has normal 2-rank ≥ 3 and admits a non-trivial nilpotent signalizer functor θ, then*

$$\Gamma_{S,2}(G) \leq N_G(\theta(E)).$$

In particular, if G is simple then G has a proper 2-generated core.

Corollary 25 *If G is a tame simple group of odd type, $n(G) \geq 3$ and $O(C_t) \neq 1$ for some involution $t \in G$, then G has a proper 2-generated core.*

Aschbacher's Component Analyses. From now on we consider groups which satisfy the B-conjecture and use some important and beautiful ideas of Aschbacher [7].

Denote by $\mathcal{E}$ the set of all infinite components of the centralizers of involutions in G.

Theorem 26 (Borovik [15]) *Assume that a tame simple K^*-group G of odd type satisfies the B-conjecture and $n(G) \geq 3$. The either $\mathcal{E} \neq \emptyset$ or G has a proper 2-generated core.*

In the proof of Theorem 26 we used the following analogue of a theorem by Ali Asar [5] for locally finite groups. Recall that a *four-group* is an elementary abelian group of order 4.

Theorem 27 *Let S be a Sylow 2-subgroup in a simple tame K^*-group G of odd type and $T = S^\circ$ the maximal 2-torus in S. Assume that $T \neq 1$. If S contains a four-subgroup V with the property that $T \trianglelefteq C_v$ for all involutions $v \in V$, then G has a proper 2-generated core.*

Classical involutions. Let t be an involution in G. A component $A \trianglelefteq E(C_t)$ is called *intrinsic* if $t \in Z(A)$. Following Aschbacher [8], we call an involution *classical* if its centralizer contains an intrinsic component isomorphic to $\mathrm{SL}_2(K)$ for an algebraically closed field K.

The role of classical involutions is shown by the following result. Its proof was adapted from Walter [45].

Theorem 28 (Borovik [15]) *Assume that G is a simple tame K^*-group of odd type and satisfies the B-conjecture. If $\mathcal{E} \neq \emptyset$ then either $n(G) \leq 2$, or G has a proper 2-generated core, or G possesses a classical involution.*

Main theorem. Now, summarising our arguments, we can state the main result of the paper.

Theorem 29 (Borovik [15]) *Let G be a simple tame K^*-group of finite Morley rank and odd type. Then one of the following statements is true.*

(i) *$n(G) \leq 2$.*

(ii) *G has a proper 2-generated core.*

(iii) *G satisfies the B-conjecture and contains a classical involution.*

Recall that in Finite Group Theory, the finite simple groups satisfying clauses (i), (ii), (iii) of the above theorem were completely classified by Gorenstein and Harada [28], Aschbacher [6] and Aschbacher [8], correspondingly. In the context of finite Morley rank there is a considerable progress in cases (i) and (ii). Indeed, in case (i) (probably the most difficult of all three) one has to classify "relatively small" groups of odd type; the recent works by Delahan and Nesin [23], Epstein and Nesin [24], and Nesin [40] contain very important results concerning this situation.

We hope that there are no simple tame K*-groups of odd type satisfying case (ii). The notion of a proper 2-generated core is very close to that one of a *strongly embedded subgroup* (cf. [17]). Tame K*-groups of *even type* with a strongly embedded subgroup were classified by Altınel [1], who used methods of a very important and difficult work by DeBonis and Nesin [22]. We believe that approximately the same ideas will work for groups of odd type.

Case (iii) has been recently solved by Ayşe Berkman; we hope that she will speak on it at the Euroconference *Groups of Finite Morley Rank*, Crete, 22–26 June 1998.

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