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What is It That Makes a Mathematician?

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Introduction: what is the purpose of this text?

Our meeting *Where will the next generation of UK mathematicians come from?* will concentrate on the education policy issues arising from our desire to nurture future mathematical talent. However, a brief look at the programme of the meeting shows that no discussion of *what mathematical abilities and talent are* is scheduled. I hope that we have a shared understanding sufficient for a meaningful conversation. Nevertheless I believe that some coffee break chats about the nature of mathematical abilities and their early manifestations in children might be useful. To facilitate an informal discussion of a highly elusive topic, I have decided to offer my notes on mathematical thinking for the attention of the participants of the meeting.

At this point, a disclaimer is necessary. I emphasise that I am not a psychologist nor a specialist in educational theory. My notes are highly personal and very subjective. They do not represent results of any systematic study. The notes are mostly based on my teaching experience in Russia in the 1970s and 1980s, in a social and cultural environment very distant from the modern British landscape.

If so, why did I bother to write this text? I teach at a university; I am concerned that our mathematics students frequently lack (and are not being trained in) specific cognitive skills which are crucially important for the profession. But I believe that these very skills (although "traits" is a better word since they might be still undeveloped) can be found in able children at as early stages of education as pre-GCSE.

> *I believe that mathematical cognitive traits should be supported and developed as soon as they first appear in a child.*

If I formulate my views in a few words, I believe in the unity of mathematics, including its *vertical unity*. For me, "recreational", "elementary", "undergraduate" and "research" mathematics are no more than artificial subdivisions of a single continuous spectrum. Writing from the position of a university teacher and PhD supervisor, I freely move through the whole range – but here my emphasis is firmly on the early stages of school mathematics.

I believe that university mathematicians should be concerned more about mathematics teaching as a system, from primary school to A levels to PhD studies (and do not forget such crucially important branches as teacher training or the teaching of mathematics to engineers). The ultimate aim of this text is to help to persuade my university colleagues that they have to start helping schools in some serious way – for example, by helping to run enrichment and extension activities.

The notes are not as comprehensive as I may wish. I try not to duplicate the existing literature and concentrate on those aspects of mathematical thinking which are not covered in the classical work by Vadim Krutetskii [**6**] or in the seminal book by Hadamard [**5**].

1. Flies and elephants

A tacit rite of passage for the mathematician
is the first sleepless night caused by an unsolved problem.

B. Reznick [**10**]; quoted from T. Gardiner [**2**]

To warn about difficulties involved in the recruitment of future mathematicians, I start with a parable which might look excessively clinical.

During World War II, Sub Lieutenant Zasetsky received a severe head wound which resulted in persistent brain damage. He was observed over 23 years by Professor Luria who wrote his famous book [**7**] based on Zasetsky's diaries (the latter comprise more than 3000 pages). Yuri Manin, when discussing the nature of proofs in his book *Provable and Unprovable* [**8**], quotes some really astonishing fragments of Zasetsky's diaries:

> And more: "is the elephant larger than the fly" or "is the fly larger than the elephant". I understood only that "the fly" is small and "the elephant" is big, but, for some reason, could not find my way through the words and answer the question, is the fly smaller than the elephant, or is it larger. The main trouble was that I could not understand what the words "is larger" refer to – the fly or the elephant.

Discussing this fragment, Manin stresses the complexity of the metalanguage text which describes the faults in the understanding of the primary language. In that particular instance, it could be possibly explained by the fact that Zasetsky is talking about the past. But here is an excerpt written in the present tense:

> . . . I again try to recall the meaning of the expressions "the fly is smaller than the elephant" and "the fly is larger than the elephant". I try to think about them, what is the correct way to understand them and what is incorrect. If we permute the words in these expressions, they change their meaning. But they look the same to me, as if nothing changed after the words were swapped. But if you think a bit longer, you notice that permutation changes the meaning of these four words (elephant, fly, smaller, larger). But my brain, my memory after I got my wound, and even now, cannot immediately grasp, what the word "smaller" (or "larger") refers to—to the elephant or to the fly. Even in these four words, there are too many permutations.

Manin uses Zasetsky's tortured account to refute Russell's thesis that even a moron should be able to check the validity of a formal proof presented as a sequence of mechanical inferences. Manin comments that, to the contrary, humans are useless at checking formal proofs. There is a clear difference between higher level reasoning and lower level verification and acceptance of elementary facts ("the fly is smaller than the elephant"). We are useless at checking formal proofs because we actively dislike to use our higher level reasoning facilities for routine actions which should normally be done subconsciously. Our capacity for higher level reasoning is so precious a resource of our mind because it is so scarce: Zasetsky was trying to resolve by conscious and controlled reasoning (information processing rate: about 16 bits per second) a problem which is normally handled by the visual processing modules of our brain (information processing rate: 10,000,000 bits per second).[1]

I draw two lessons from Zasetsky's account.

First, when teaching mathematics, we have to remember this miserable number: 16 bits per second for conscious information processing (which is further reduced to 12 bits per second for multiplication of numbers or 3 bits per second for counting objects). Our students will not master a mathematical technique or concept unless much more powerful mechanisms of subconsciousness are engaged. Just compare these two numbers: 16 and 10,000,000!

The second lesson is about the emotional side of mathematics. My fellow mathematician, do you recognise yourself in Zasetsky's self-portrait?

I do.

It so happened, that half an hour before I red Manin's book, I spent some time in a conversation with a colleague trying to figure out whether a certain matrix corresponded to a linear map $U \to V$ or to the map of dual spaces $U^* \to V^*$ (in the context where we have already switched several times, forth and back, between spaces and their duals, the issue was not so much difficult—it definitely was not—as it was highly confusing). We were in a typical "the fly and the elephant" situation; this is why reading—just minutes later—Zasetsky's confessions was like a shock to me. Only after my colleague and I used, unsuccessfully, every trick to resolve the issue at the conceptual or intuitive level, did we resort to a formal calculation on paper, which, of course, gave us the answer. But remarkable was our very reticence to do the formal calculation; instead, we were seeking ways of making the choice self-evident, because we felt that it would turn out to be more valuable to us. Indeed, a calculation establishes the fact and its result can be formally reused. On the other hand, making a fact self-evident does not establish its formal validity; it still requires a proof. However, self-evident things can be reused, at the intuitive level, in further mental work (I avoid the term "reasoning" here), they will jump up, at the right times, from the subconscious levels of our mind into the areas controlled by conscious reasoning.

I trust that my fellow mathematicians would also agree that Zasetsky's accounts of his mental torture can be used as an explanation, to a non-mathematician, why mathematics as a professional occupation is so uncomfortable. (But why should we explain that? Perhaps it is better hide the unpalatable truth.) Mathematicians are sometimes described as living in an ideal world of beauty and harmony. Instead, our world is torn apart by inconsistencies, plagued by *non sequitur*, and worst of all, made desolate and empty by missing links between words, and between symbols and their referents; we spend our lives patching and repairing it. Only when the last crack disappears, are we rewarded by brief moments of harmony and joy.

And what do we do then? We start to work on a new problem, descending again into chaos and mental pain. We do that to earn the next fix of elation.

Maybe this truth is not for public consumption, but many (and some of the brightest) mathematicians are "problem-solving" analogues of gambling addicts and adrenalin junkies. My best PhD student once complained to me that she was exhausted, because for two weeks, she awoke every morning with a clear realisation that she continued to think about a problem in her sleep. She was a *real* mathematician. Where can we find more students like her?

2. Some theory: reification and encapsulation

To make my personal observations closer to the established methodological framework of mathematics education theory, I wish to turn to the discussion of reification.

The term *reification* was introduced into mathematics education studies by Anna Sfard, who applied it to the process of objectivisation of mathematical activities. There is a significant body of literature, both theoretical and experimental studies, which deals with reification mostly in the framework of school mathematics teaching. The concept is pretty close to that of *encapsulation* [**16**]. One may wish to find subtle differences in the meaning of the two concepts, but, since in application to real case studies they become blurred anyway, I do not see the need to take the possible difference into account.

The associated verb is *to reify*, [2] with the meaning "to convert mentally into a thing, to materialise". This concept is exceptionally useful in the understanding of mathematics teaching and learning. According to Reuben Hersh's succinct description, children first learn an activity, something they do; this activity is frequently formalised as an algorithm, but sometimes remains semiformal. Later the activity becomes a "thing", something they can think about as an object. This "reification" step is difficult for a student (see its discussion in the dialog between Anna Sfard and Pat Thompson [**12**]) and is the main contributing factor to the success or failure of mathematics teaching.

The term "encapsulation" is somewhat more convenient because it allows us to define a natural opposite action, *de-encapsulation*; "de-reification" sounds odd. This is the description of encapsulation and de-encapsulation in Weller et al. [**16**, p. 744]:

> The encapsulation and de-encapsulation of processes in order to perform actions is a common experience in mathematical thinking. For example, one might wish to add two functions f and g to obtain a new function $f + g$. Thinking about doing this requires that the two original functions and the resulting function are conceived as objects. The transformation is imagined by de-encapsulating back to the two underlying processes and coordinating them by thinking about all of the elements x of the domain and all of the individual transformations $f(x)$ and $g(x)$ at one time so as to obtain, by adding, the new process, which consists of transforming each x to $f(x) + g(x)$. This new process is then encapsulated to obtain the new function $f + g$.

It is instructive to see how Anna Sfard assesses a mathematician's description of his work. In [**12**] she quotes a famous mathematician, Bill Thurston:

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics. [**14**, p. 847]

Sfard comments on this:

> If the "compression" is construed as an act of reification - as a transition from operational (process-oriented) to structural vision of a concept . . . , this short passage brings in full relief the most important aspects of such transition. First, it confirms the developmental precedence of the operational conception over the structural: we get acquainted with the mathematical process first, and we arrive at a structural conception only later. Second, it shows how much good reification does to your understanding of concepts and to your ability to deal with them; or, to put it differently, it shows the sudden insight which comes with "putting the helmet and glove on"[3] with the ability to see objects that are manipulated in addition to the movements that are performed. Third, it shows that reification often arrives only after a long struggle. And struggle it is!

I would not construe Thurston's words in the same way. What he describes is *not* reification. More precisely, reification is present in the process, but makes only a tiny portion of it. Even where reification is present, it is directed by mathematical structures, by metatheories, and is quite purposeful and intentional. The process of compression, as described by Thurston, also involves a systematic search for new languages or translation of the problem into other known languages; meta-arguments and analysis of existing proofs, etc. To call all these actions reification means to stretch the useful concept to the point when it becomes all-embracing and vacuous. Moreover, the reification itself is frequently compressed, the same way as other mathematical activities tend to compress themselves into reusable units.

In Vadim Krutetskii's classical study of psychology of mathematical abilities in children "compression" appears under the term of "curtailment of the reasoning process". It is interesting that he documented a very ambivalent attitude of teachers to schoolchildren who demonstrate the ability to "curtail" mathematical reasoning [**6**, p. 189]. On one hand, some teachers see it as an important manifestation of mathematical abilities:

> "In capable pupils the reasoning process is curtailed and is never developed to its full logical structure. This is very economical, and in this lies its value."
>
> "I have often observed how a capable pupil thinks: for the teacher and the class it is a detailed process, with all the links in the sequence, and for himself it is fragmentary, cursory, very abbreviated, a shorthand record of thought."

But some teachers were apparently at loss:

> "I do not know how to evaluate this. In school we persist in teaching pupils to give the complete logical argument."
>
> "I think that the mathematically able mind is a clear, deliberate, logical mind, and any acceleration or abbreviation should be alien to it. Logic is logic; you never get away from it. To leave out a link would not be logic."

Krutetskii's study was done almost 50 years ago, in a social and cultural environment very distant from the modern British educational landscape. It would be interesting to analyze the responses of British schoolteachers of today. But I feel that the issue of the role of reification and encapsulation in teaching and learning mathematics is far from being resolved.

3. Reification: a small case study

I wish to start a brief (and very incomplete) list of mental traits necessary for a working mathematician. Of course, obsessive persistence, Zasetsky style, to retie the torn bonds between concepts should feature prominently on any such list.

Here, I wish to add another item to the list of a mathematician's cognitive traits:

> A mathematician is someone who reifies abstract concepts intentionally and purposefully, and who can reuse, in compressed form, the psychological experience of previous reifications.

It is my conjecture that potential future mathematicians are boys and girls who, at the age when their classmates struggle to reify the concept of a linear equation (nothing derogatory in my comment: indeed, it is a difficult concept), can already reify at will (of course, within the limits of mathematics they know).

A brief case study will possibly be useful. Here is a problem I liked to give at the selection interviews for the Novosibirsk Summer School (the penultimate step of the selection procedure of Fizmatshkola, the Preparatory Boarding School of Novosibirsk University):

> Given 2004 distinct points on the plane, prove that there exists a straight line which divides the points in two groups of 1002 points each.

I encouraged my interviewees (14–15 years old boys and girls) to talk about any ideas they could propose towards the solution; I watched attentively for any signs of understanding, on their part, that

(a) the fact deserves a rigorous proof and can be proven; and
(b) the words "there exists" in the problem are likely to mean an invitation to produce an explicit procedure for constructing such a line.

There is at least one simple solution: draw lines through each pair of points; take a line far away from the points and such that it is not parallel to any of the lines through pairs of points; move this new line towards the points, keeping it parallel to its original position. In that way, the moving line will meet the points one by one, thus allowing for counting. All solutions actually produced by children involved similar counting procedures, with lines rotating, circles expanding, etc.

The reader would probably agree that what is required here is the ability to think about the procedure as a single entity, as an object, specifying first the list of requirements for the procedure; and to do this as a one-off problem (most likely, my interviewees had never before in their lives encountered problems in any way similar to that one), without the guiding hand of the teacher, without a long series of preparatory exercises.

Reification is difficult; as Anna Sfard describes it [**12**],

> The main source of this inherent difficulty is what I once called the (vicious) circle of reification – an apparent discrepancy between two conditions which seem necessary for a new mathematical object to be born. On one hand, reification should precede any mention of higher-level manipulations on the concept in question. Indeed, as long as a lower-level object (e.g. a function) is not available, the higher-level process (e.g. combining functions) cannot be performed for the lack of an input. On the other hand, before a real need arises for regarding the lower-level process (here: the computational procedure underlying the

> function) as legitimate objects, the student may lack the motivation for constructing the new intangible "thing." Thus, higher-level processes are a precondition for a lower-level reification – and vice versa!

Therefore I add to my list of mathematician's traits:

A mathematician recognises the vicious circle of reification and actively seeks ways to break it.

Surprisingly, the general populace contains a number of children who had somehow developed this ability. It is worth mentioning that, in the selection to *Fizmatshkola*, the interviewers were instructed never to ask questions about children's academic performance at school; the standard interview form filled in at each interview contained no fields for the interviewee's school grades. However, we dutifully collected the names of theirs mathematics teachers. As you might expect, a small number of teachers produced a disproportionate number of able students. What always interested me was how these teachers taught; what made their students so special? How can the skills of *reification on demand* be taught? Until such time as a better national system is in place, one way of increasing the number of potential mathematicians might be to identify precisely this kind of teacher, to circulate a newsletter among them, and to arrange regional meetings to encourage them see themselves as an informal "vanguard". Or would this undermine their quiet effectiveness?

4. Multiple representation and de-encapsulation

The starting point of my next small case study is another excerpt from the dialog of Thompson and Sfard on the nature of reification [**12**].

Thompson quotes his earlier paper [**13**, pp. 39]:

> I believe that the idea of multiple representations, as currently construed, has not been carefully thought out, and the primary construct needing explication is the very idea of representation. Tables, graphs, and expressions might be multiple representations of functions for us, but I have seen no evidence that they are multiple representations of anything to students. In fact, I am now unconvinced that they are multiple representations even to us…

Pat Thompson later adds that that background motivation for this statement "was largely pedagogical". On the contrary, Sfard sees the whole point in that

> being able to make smooth transitions between different representations […] means there is something that unifies these representations.

What Anna Sfard calls a "mathematical object" is such a unifying entity; for her, mathematical objects are reified mathematical processes, the unifying entities of something the learner of mathematics has already done.

Similarly to what I said in Section 3, the crucial difference between a mathematician and a novice learner of mathematics is that

> A mathematician actively seeks *new* or known, but previously ignored representations and interpretations of his or her mental objects.

And here is another problem which I used in my mathematical interviews.

> Some anglers caught some fish. It is known that no-one caught more than 20 fish; that a_1 anglers caught at least 1 fish, a_2 anglers caught at least 2 fish, and so on, with a_{20} anglers catching 20 fish. How many fish did the anglers catch between them? (Of course, in more concrete versions of the problem $\{a_i\}$ can be replaced by any non-increasing sequence of non-negative integers.)

I remember two principal types of solutions.

Solution 1. Notice that exactly $a_{19} - a_{20}$ anglers caught 19 fish, $a_{18} - a_{19}$ anglers caught 18 fish, and so on. Therefore the total number of fish is

$$20a_{20} + 19(a_{19} - a_{20}) + 18(a_{18} - a_{19}) + \cdots + 2(a_2 - a_3) + (a_1 - a_2),$$

which simplifies to

$$a_1 + a_2 + \cdots + a_{19} + a_{20}.$$

As you can see, some skills of formal manipulation with sequences would be quite handy.

Solution 2. This solution is more interesting in the context of "multiple representation", since it involves, first, a clear understanding of the concept of functional dependence on the part of the solver, and second, the preparedness to look at one of the most primitive forms of representation of functional dependence: charts.

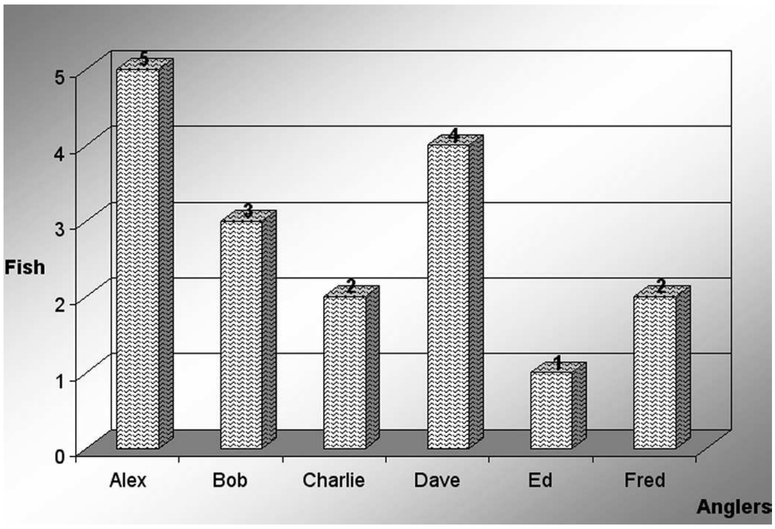

FIGURE 1. Some anglers caught some fish, a chart made by MICROSOFT EXCEL.

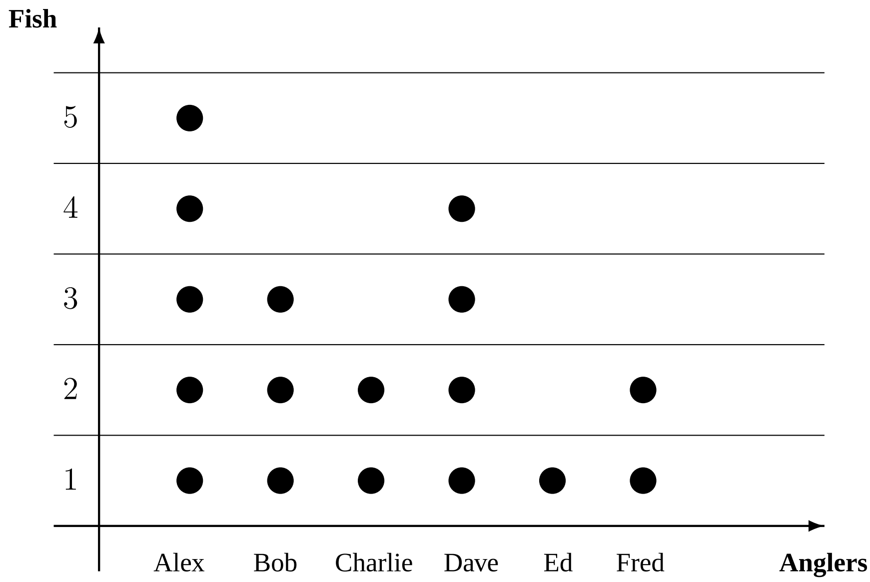

FIGURE 2. Some anglers caught some fish, represented by dots on graphed paper.

The chart in Figure 1 is the best I could squeeze from MICROSOFT EXCEL after 15 minutes of trying, and is a good example of why software based learning of mathematics is intrinsically flawed: the software forces on you the mode of visualisation. However, visualisation is too intimate a component of mathematical thinking to be entrusted to a computer. Instead, let us draw a simple diagram representing fish by thick black dots on graphed paper, see Figure 2. In this much more primitive chart, a_i is the number of dots in row i, which immediately gives the total number of dots as $a_1 + a_2 + \cdots$.

Some lessons to be learned from this small case study:

- The second solution requires some reasonable level of handling of a general concept of functions (of nominal, not numeric, variables: the arguments of the function are names, not numbers!).
- However, the solver should be prepared to choose a very low level concrete representation of the general concept; one may wish to say, using terminology of [**16**], de-encapsulate it down to a rather primitive level. From the teacher's perspective, this means that earlier, lower level material should be not just well understood by a student, it should be absorbed, interiorised to the point of totally automatic, subconscious use. De-encapsulation is no less important than encapsulation; the student has mastered the encapsulated concept only if she can de-encapsulate it at will, and freely choose the most appropriate of many possible modes of de-encapsulation.

- The solver has to actively probe his mind for various representations of the problem (or translations to various mathematical languages) while the most appropriate one is found.
- Finally, the problem itself is not that naive: its solution with dots is a miniature version of the Fubini Theorem for the Lebesgue integral. I bet the mathematician who originally set the problem knew this connection, and in the most explicit terms.
- The first solution of the problem requires a higher level of symbolic mathematical technique. Again, it is remarkable that the solution reflects, at a very elementary level, some non-trivial mathematical results.[4]

Many eloquent speeches were made, and many beautiful books written in explanation and praise of the incomprehensible unity of mathematics. In most cases, the unity was described as a cross disciplinary interaction, with the same ideas being fruitful in seemingly different mathematical disciplines, and the technique of one discipline being applied to another. The *vertical* unity of mathematics, with many simple ideas and tricks working both at the most elementary and at rather sophisticated levels, is not so frequently discussed—although it appears to be highly relevant to the very essence of mathematics education.

5. The economy principle

The following informal concepts of mathematical practice cry out to be explicated:

beautiful, natural, deep, trivial, "right", difficult, genuinely, explanatory ...

Timothy Gowers

Quite a number of phenomena of mathematical practice can be explained in terms of what I call, for the lack of better name, the *economy principle*:

> A mathematician has an instinctive tendency to favor objects, processes and rules with the simplest possible descriptions or formulations.

To some extent it is a general tendency of the human mind; it is taken for granted, for example, by composers of IQ tests, where answers to various problems of the type

...continue the following sequence:

$$1, 3, 6, 10, 15,$$

(this one is taken from one of the numerous IQ test websites) are expected to be based on the assumptions (which incidentally are never stated) that

(a) the numbers or objects in the sequence are supposed to be built consecutively one by one, and
(b) the rule for construction has to be as simple as possible.

In that particular case, one can easily observe that consecutive increments in the sequence are 2, 3, 4, 5 and therefore a likely continuation of the sequence is

$$1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28\ldots$$

IQ test number sequences provide some of the best mathematical entertainment on the Internet: use GOOGLE to find an IQ test, copy a sequence and paste it in the search engine of the *On-Line Encyclopedia of Integer Sequences*[5]. Then look in awe at the astonishing number of mathematically meaningful descriptions and ways to continue the sequence. Notice that each of the sequences has actually appeared in some mathematical problem – *Encyclopedia* provides comprehensive references! For example, one of the ways to continue our sequence 1, 3, 6, 10, 15 is

$$1,\ 3,\ 6,\ 10,\ 15,\ 20,\ 27,\ 34,\ 42\ldots$$

(sequence A047800). It has the explanation that its n-th term is the number of different values of $i^2 + j^2$ for i and j running through the integers in the interval $[0, n]$.

However, we instinctively know that the first answer is "right" because its description is simpler.

We can easily see that the "economy principle" is an important part of many informal concepts of mathematical practice. For example, Timothy Gowers once observed that, in his opinion, a "comprehensible" proof is not necessarily the shortest one, but a proof of small *width*. Here, width measures how much you must hold in your head at any one time. Alternatively, imagine that you write a detailed proof on a blackboard, carefully referring to all intermediate steps. However, if you know that a certain formula or lemma will never be used again, you erase it and re-use the space. A "small width" proof is a proof which never expands beyond one (small) blackboard.

Vadim Krutetskii emphasises that striving for clarity, simplicity and economy in a solution is one of the most important signs of mathematical abilities in children. I quote two examples from his work [**6**].

In the first example [**6**, p. 285], S. G., an eighth grader (that is, 14 or 15 years old), solves the following problem:

> (*Problem XIX-A-11* of [**6**]) Find a four-digit number with the following conditions: the product of the extreme digits is equal to 40; the product of the middle digits is 28; the thousands digit is as much less than the units digit as the hundreds digit is less than the tens digit; and if $3,267$ is added to the unknown number, the digits of the number are reversed.

It is interesting that S. G. made an explicit choice between two strategies:

> Initially she composed a complex system of equations in four unknowns (the way almost all pupils began). Without trying to solve the system she composed, S.G. said: "This can be solved but it's very awkward. There ought to be a simpler solution here somewhere. But equations aren't needed here: 40 can be the product of just two numbers: $5 \cdot 8$. But the thousands digit is less than the units digit, and then the number is like this: $5 * * 8$. Well, this is clear. The number is 5,478."

At this point it also becomes clear why "curtailment of thinking" can sometimes be irritating to teachers (see p. 5). When pressed for explanation, S. G. clarified:

> "28 is the product of only two digits: $4 \cdot 7$. The hundreds digit is less than the tens digit. These digits only have to be arranged."

An idea for handling an intermediate step, the uniqueness of factorisation of 40 into digits, $40 = 5{\cdot}8$, was immediately re-used by S. G. as something obvious and not deserving further mentioning.

In the second example [**6**, p. 284], the interviewee is 9-year-old Sonya L.

> *Problem.* A father and his son are workers, and they walk from home to the plant. The farther covers the distance in 40 minutes, the son in 30 minutes. In how many minutes will the son overtake the father if the latter leaves home 5 minutes earlier than the son?
>
> *Usual method of solution [by 12-13 year old children]:* In 1 minute the father covers $1/40$ of the way, the son $1/30$. The difference in their speed is $1/120$. In 5 minutes the farther covers $1/8$ of the distance. The son will overtake him in
> $$\frac{1}{8} : \frac{1}{120} = 15 \text{ minutes.}$$
> *Sonya's solution:* "The father left 5 minutes earlier than the son; therefore he will arrive 5 minutes later. Then the son will overtake him at exactly halfway, that is, in 15 minutes."

6. Hidden symmetries

Krutetskii's case studies, sharply observed and precisely recorded, frequently contain more than he chooses to highlight and comment on. Actually, in the previous example Sonya L. used a trick which deserves to be specifically mentioned: she noticed a hidden symmetry in the set up of the problem and immediately exploited it.

> A mathematician seeks and exploits hidden symmetries in a problem.

The word "symmetry" here has to be understood in the most wide interpretation and applied not only to geometric symmetry as we know it, but also to "semantic" or "logical" symmetry.

The famous *Pons Asinorum* theorem[6] of Euclidean geometry provides a very poignant example. The theorem says that

> the base angles of an isosceles triangle are equal.

My own teaching experience (back in Russia in the 1980s) showed that surprisingly many students were able to see that *Pons Asinorum* could be proven by a direct argument based on the *formal* symmetry of the premises:

- $AB = AC$
- $AC = AB$
- $\angle BAC = \angle CAB$ (since the angle is equal to itself).
- $\triangle BAC = \triangle CAB$ (Side-Angle-Side criterion of congruence).
- Therefore, $\angle B = \angle C$.

The proof in the school textbooks was, of course, different because the "formal symmetry" proof was deemed to be too difficult for schoolchildren (and it probably was for many of them). Instead, the *Pons Asinorum* was proven outside the axiomatic system, by a direct application of the bilateral symmetry of the triangle viewed as a cardboard cutting.[7]

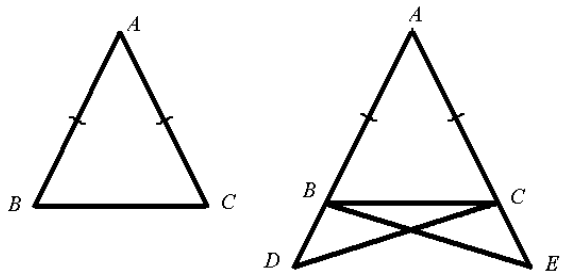

FIGURE 3. *Pons Asinorum*. Euclid proved the theorem in a complicated way, by extending the sides and making the exterior angles $\angle DBC$ and $\angle ECB$ to belong to two different but congruent triangles. In his proof, we choose points B and C so that $BD = CE$. Then we show that $\triangle DAC = \triangle EAB$ and at the next step that $\triangle CBD = \triangle BCE$. So $\angle CBD = \angle BCE$. After that $\angle CBA = \angle BCA$ as exterior angles to equal angles.

The dispute about the usability of the "formal symmetry" proof in teaching apparently has a long and honorable history – starting with Euclid himself, who did not use it in his *Elements*. With the advent of computers and Artificial Intelligence the story found a fascinating turn – proofs based on "semantic symmetry" (the term is from [**3**]) had happened to be natural for automated proof systems. The first breakthrough was made by famous computer scientist Marvin Minsky: in 1956, his (hand-simulated) program easily found the "formal symmetry" proof of *Pons Asinorum*. A comment by Minsky is quite revealing:

> What was interesting is that this was found after a very short search - because, after all, there weren't many things to do. You might say the program was too stupid to do what a person might do, that is, think, "Oh, those are both the same triangle. Surely no good could come from giving it two different names."[8]

As a side comment, I wish to express my regret that proofs are increasingly suppressed in mathematics teaching. I can say from my teaching experience at Manchester that students from Greece and Cyprus handle the concept of proof much better than British students. Euclid happened to be Greek, and, as a matter of national pride, Euclidean geometry is still being taught in Greek schools.

Last comments

I can easily continue the list of specific traits of a mathematicians which manifest themselves at relatively early stages of learning mathematics.

For example, one can observe that one of the most natural ways of encapsulating complex processes is treating them as games. (For otherwise why do children, in all cultures, play games?) But, in many cases, a mathematician has to figure out the rules of the game as he plays it.

The game context makes natural *abstraction by irrelevance* – and children could be quite good at it. Lines have no width not because we want them so, but because we do not care about the width – we are using them in a situation where width does not matter.

However, I stop here in the hope that the present text suffices for starting a dialogue. I would appreciate any comments, corrections, criticism.

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The gods have imposed upon my writing
the yoke of a foreign tongue
that was not sung at my cradle.
Hermann Weyl

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Notes

[1]See a fascinating discussion of "bandwidth of consciousness", with references to the original psychological research, in Tor Nørretranders' book [**9**]. The bit rate tables are on pp. 138 and 143.

[2]In the marxist literature, the term "reification", as well as its more specialised version, *commodisation*, has rather negative connotations, which are absent in Sfard's use of the word.

[3]Sfard uses here her "virtual reality game" metaphor of mathematics: you don a helmet with visors, a glove with motor sensors and suddenly see a world where you can move objects [**11**].

[4]Indeed, let us try to find a calculus version of the formula

$$\text{Fish total} = 20a_{20} + 19(a_{19} - a_{20}) + \cdots + 2(a_2 - a_3) + (a_1 - a_2).$$

To avoid the use of Lebesgue measure, assume that anglers are points on $[0, 1]$, and that the number $y = y(x)$ of fish caught by angler x is a differentiable strictly decreasing function. Then the inverse function $x = x(y)$ conveniently represents the number of anglers who caught at least y fish. The formula for the total number of fish becomes the identity

$$\int_0^1 y dx = \int_{y(0)}^{y(1)} y \cdot \frac{dx}{dy} \cdot dy.$$

[5]*On-Line Encyclopedia of Integer Sequences*, http://www.research.att.com/∼njas/sequences/Seis.html, is a brainchild of N. J. A. Sloane. A fantastic resource, it simply begs to be used in some enrichment activity for school mathematics.

[6]Coxeter writes in [**1**]: "The name *Pons Asinorum* for this famous theorem probably arose from the bridgelike appearance of Euclid's figure (with the construction lines required in his rather complicated proof) and from the notion that anyone unable to cross this bridge must be an ass. Fortunately a far simpler proof was supplied by Pappus of Alexandria about 340 A.D." [And that was exactly the "formal symmetry" proof which I discuss in this paper.– AB]

[7]This approach was promoted by Hadamard in his highly influential *Leçons de géométrie élementaire* [**5**] and adopted by canonical Russian textbooks.

[8]Quoted from http://www.math.niu.edu/ rusin/known-math/99/minsky.

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