Hidden structures of elementary mathematics

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http://www.ams.org/bookstore-getitem/item=mbk-71
1. Little green men from Mars

If the little green men from Mars existed, would their mathematics be the same as ours?
The same question restated in the ethnographic context:

- Pirahā people of South America have no concept of numbers whatsoever (Frank et al. 2008).
• Tenejapan Tzeltal people of Central America have no words or concepts for the left and the right—they live in square houses, with a door (made of two equal halves) positioned in the middle of the south wall (Brown 1992).
• Guugu Yimithirr people of Australia live in the absolute (and Cartesian!) coordinate system.

When asked “where is the kettle?”, they answer “two steps north of the fireplace” (Haviland 1998).

Which mathematics is more natural for them: Euclidean or Cartesian?

For a mathematician, there is no much difference; for a maths educationalist, the difference is huge!
• some tribes of Equatorial Africa have no awareness of geometric perspective (Deregowski 1974).

Let’s pause; this comes closest to little green men from Mars.
Peters’ Elephantnose Fish (*Gnathonemus petersii*) in the Mathematics Department aquarium, University of Manchester.

Distribution of charges along the body creates electromagnetic fields used for identifying prey and communication.
Elephantnose Fish: its image of the world has infinite dimensional group of symmetries.
Conformal maps in medical imaging: a standardised image of the surface of a human brain is a conformal map onto a sphere.

Conformal transformations allow for infinitely many “degrees of freedom”.
Hans Holbein, The Ambassadors, 1533.
The skull with fixed perspective.

Human eye has to deal with the group of projective transformations $\text{PGL}_3(\mathbb{R})$ which has very modest 8 degrees of freedom.

It is already too much for our brain.
Whatever makes a Deity without the Knowledge of Perspective will be liable to such Mischance as are shewn in the Woodcut.
W. Hogarth (1697–1764), “A satire on false perspective” (1753). ‘Whoever makes a Design without the knowledge of Perspective will be liable to such absurdities as are shewn in this Frontispiece.”

Recalculation of projective images is done **locally**.
Another forgotten tribe of humanity:

- children.
A child in Zimbabwe, 1980s, pushing a wire toy automobile. (After Deregowski)
A wire toy pedal vehicle.
A wire toy automobile.
A case study:

Why does the mirror changes left and right

...but does not change up and down?
Why does the mirror changes left and right ... but does not change up and down?

**Answer:** it does not.

Instead it changes forward/backward directions.

Changing of left and right is just a popular myth.
Why does the mirror changes left and right . . . but does not change up and down?

**Answer:** it does not.

Changing of left and right is just a popular myth.

**Question:** Why does the myth exist?
Why does the mirror change left and right... but does not change up and down?

**Answer:** it does not.

Changing of left and right is just a popular myth.

**Question:** Why does the myth exist?

**Answer:** Because we attribute to the mirror the intrinsic bilateral symmetry of our mind.
Blaise Pascal:

Our notion of symmetry is derived from the human face. Hence, we demand symmetry horizontally and in breadth only, not vertically nor in depth.
Immanuel Kant:

In physical space, on account of its three dimensions, we can conceive three planes which intersect one another at right angles. Since through the senses we know that which is outside us only in so far as it stands in relation to ourselves, it is not surprising that in the relationship of these intersecting planes to our body we find the first ground from which to derive the concept of regions in space . . .

One of these vertical planes divides the body into two outwardly similar parts and supplies the ground for the distinction between right and left; the other, which is perpendicular to it, makes it possible for us to have the concept before and behind.
It is symmetry of mind that matters.
Mirror writing

1 in 6000 people can write by non-dominant hand in mirror script.

The rest 5999 can do a weaker version of mirror writing.
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The rest 5999 can do a weaker version of mirror writing:

Mirror reflection:
Mirror writing: Leonardo da Vinci

Everyone know his famous study of the symmetry of human body:
It is less known that Leonardo’s notes are in mirror writing:
Mathematical illustration: Euler’s Theorem

If an orientation-preserving isometry of the affine Euclidean space $\mathbb{A}\mathbb{R}^3$ has a fixed point then it is a rotation around some axis.

Numerous psychological experiments show that Euler’s Theorem is hardwired in our brain.
Coxeter's proof, verbatim:

In three dimensions, a congruent transformation that leaves a point \( O \) invariant is the product of at most three reflections: one to bring together the two \( x \)-axes, another for the \( y \)-axes, and a third (if necessary) for the \( z \)-axes.

Since the product of three reflections is opposite, a direct transformation with an invariant point \( O \) can only be the product of reflections in two planes through \( O \), i.e., a rotation.
The intuition of symmetry is rooted in both visual and sensorimotor systems.

Another example of such overlap: convexity.

A symptom of interaction between sensorimotor and visual intuitions: **proof by handwaving**.
Grinding flat mirrors

Take three pieces of glass and grind 1-st and 2-nd pieces together. Then do the same for the 2-nd and the 3-rd pieces and then for the 3-rd and 1-st pieces.

Repeat many times and all three pieces of glass will become very accurately flat. Why?
A rather deep mathematics is just one step away:

What surface do we get if we grind only two pieces of glass?
A spherical stone vase, Ancient Egypt. © Petrie Museum, UCL.

The vase was made by polishing a piece of stone. Its spherical shape is a consequence of a generalisation of Euler's theorem:

a more subtle classification of subgroups in the group of movements of the 3-dimensional space.
When intuition fails: the coffee cup trick and the “global” topological structure of the rotation group

\[ 1 \rightarrow \mathbb{Z}_2 \rightarrow SU_2(\mathbb{C}) \rightarrow SO_3(\mathbb{R}) \rightarrow 1 \]

Explanation: Quaternion numbers

\[ \mathbb{H} = \{ a + bi + cj + dk \} \]
Pons Asinorum:

the base angles $\angle B$ and $\angle C$ of an isosceles triangle $\triangle ABC$ are equal.

Textbook proof, 20th century (Kiselev, Hadamard):

cfolding of a cardboard cut-out.
A proof based on the formal symmetry of premises:

- \( AB = AC \)
- \( AC = AB \)
- \( \angle BAC = \angle CAB \) (since the angle is equal to itself).
- \( \triangle BAC = \triangle CAB \) (by the Side-Angle-Side criterion of congruence).
- Therefore, \( \angle B = \angle C \).
Euclid proved *Pons Asinorum* by a sophisticated auxiliary construction, creating and using not just one, but two pairs of different but congruent triangles: $\triangle DAC = \triangle EAB$ and $\triangle CBD = \triangle BCE$. 
Testimonies from former children:

**TB:** I had trouble remembering if $a : b$ means $a$ divided by $b$ or $b$ divided by $a$. There’s no way to tell. It is arbitrary. Likewise, some years earlier, remembering which side is left and which is right.

I should also mention the notation $a \mid b$, as in $a$ divides $b$. Or maybe it was $b$ divides $a$. Probably the first one. For example: $2 \mid 4$ but not $2 \mid 3$.

I learned a reliable mnemonic for left and right when learning to drive at the age of 18 (about). Before that, it was guesswork.
MP The other was the problem of reconciling the personal right / left frame of reference with the view-from-“above” frame which is necessary for north / south . . . (the latter won and I still have problems in quickly telling right from left).
**PD:** I had a short moment of bewilderment (around 14 or 15 or 16?) when I learned that the graphs of \( f(x - a) \) and \( f(nx) \) when compared to those of \( f(x) \) behave "in the opposite way" to one's expectation. I got it very quickly and easily, but I remember being surprised.
PD: And this flows into similar adult experiences:

Does the "transition matrix" transform the basis or the coordinates? (Actually, many books hide the appearance of the inverse of the transpose by suitably defining the transition matrix.)

Given a matrix of a linear map, am I writing the map between the vector spaces or between their duals? Do the transition functions of a vector bundle transform the frame or the sections?

Am I looking at the sheaf $\mathcal{O}(D)$ or $\mathcal{O}(-D)$ ($D$ is a divisor on a variety), and do its sections have a pole or zero at $D$?

This is all one and the same question and one learns to recognize it, but it is amazing how persistent it is
BB: From the time I learned matrices (age 16 or so) I cannot remember which are the columns and which are the rows. Given that the arrangement of coefficients in a linear transformation can be written equally well in a matrix in two ways, it is something that always takes me 10–15 seconds to recall even now.
Guinea pigs: the so-called “able” children
What’s weird about 1, 11, 111, 1111 etc when you square them?

1² = 1, 11² = 121. Keep on doing this with the other numbers. (If necessary use a calculator).

Solutions see page it counts up e.g. 1111111² = 1234567654321

But when you have 1,111,111,111² the answer is different. Figuring out (or using the calculator) what are the next square numbers in the pattern after 111111111²?

Solutions see page 1234567900987654321,
123456790120987654321, 12345679012320987654321,
1234567901234320987654321 and
123456790123454320987654321!

Do you notice a pattern?

From a workbook of a schoolboy DW.
\begin{align*}
1 \times 1 &= 1 \\
11 \times 11 &= 121 \\
111 \times 111 &= 12321 \\
1111 \times 1111 &= 1234321 \\
11111 \times 11111 &= 123454321 \\
111111 \times 111111 &= 12345654321 \\
1111111 \times 1111111 &= 1234567654321 \\
11111111 \times 11111111 &= 123456787654321 \\
111111111 \times 111111111 &= 12345678987654321
\end{align*}

This is what DW meant.
But the pattern breaks at the next step and you have already noticed that:

$$1111111111 \times 1111111111 = 1234567900987654321$$

The result is no longer symmetric. Why?
My writing on a whiteboard during my first meeting with DW.

I asked DW whether the symmetric pattern of results continued indefinitely.

DW instantly answered "No".
To illustrate his point, DW's wrote down, apparently from his memory, the first case when the pattern breaks.
“Good”—said I— but let us try to figure out why this is happening”, and wrote on the board:
“Yes”—said DW—“this is column multiplication”.

“And what are the sums of columns’?”

“1, 2, 3, 4, 3, 2, 1”—dictated DW to me, and I have written down the result:
me: "Will the symmetric pattern continue indefinitely?"

DW: "No – when there are 10 1's in a column, 1 is added on the left and there is no symmetry."

me: "Yes! Carries break the symmetry. But let us look at another example"–and I wrote:

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$1^n = 1$

$(1 + x)^2 = 1 + 2x + x^2$

$(1 + x + x^2)^2 = 1 + 2x + 3x^2 + 2x^3 + x^4$

**DW was intrigued and made a couple of experiments:**
It appeared from his behaviour that he was using mostly mental arithmetic, writing down the result, term by term, with pauses:

\[
\left( \frac{1 \times x^4}{1+2x^3} \right) \cdot 1 + x + x^2 + x^3 = x^4 + 2x^3 + 3x^2 + 2x + 1
\]

\[
\text{(1 + } x + x^2 + x^3)^2 = 1 + 2x + 3x^2 + 4x^3 + 3x^4 + 2x^5 + x^6
\]

DW said with obvious enthusiasm: “Yes, it is the same pattern!”
“Wonderful”—answered I—“let us see why this is happening. I’ll give you a hint: multiplication of polynomials can be written as column multiplication”, and started to write:
DW did not let me finish, grabbed the marker from my hand and insisted on doing it himself:

![Mathematical equation image]

He stopped after he barely started the second line and said very firmly: “Yes, it is like with numbers”.

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me: ‘Well – but will the pattern break down or will continue forever?’

[Pause]

DW: “No, it will not break down!”

me: ‘Why?’

DW: “Because when you add polynomials, the coefficients just add up, there are no carries.”
me: “You know, in mathematics polynomials are sometimes used to explain what is happening with numbers”.

DW: “Yes, 10 is $x$. ”
At the time of our conversation DW was 8.
DW shows specific patterns of mathematical thinking, which are alien to majority of mathematics graduates in my University.
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Speed and fluency of generalisation suggests the ability of quickly form and operated with

scales of plausibilities of abstract statements.
Systematic teaching of mathematics to very young children:

Vladimir Radzivilovsky (Russia, now Israel)
From a workbook of one of Radzivilovsky's students: differentiation of inverse trigonometric functions.
And the student herself. Her name is Avital. She is 6.
Characteristic feature of Radzivilovsky's method: direct appeal to psychophysiological substrate of mathematical intuition.

- Asking (very) young children to guess the weight and then weigh (in grams) every household item which they could fit on the scales.
- Asking children to estimate temperature (by touching water, say), and compare their feelings with a measurement by a thermometer.
- ...
Zvonkin’s students.
And again we encounter scales.

A thermometer for measuring the “hope” to get two cubes of the same colour out of a box.
One of the psychophysical pillars of mathematical intuition:

Mechanisms of cross-modal matching of scales of intensities of sensory experiences.
One of the psychophysiological pillars of mathematical intuition:

**Mechanisms of cross-modal matching of scales of intensities of sensory experiences.**

Conjecture: the same mechanism matches scales of plausibilities of mathematical statements.
Steven's Power Law revisited?

What interests me is

the nature of mechanisms which allow us to handle one scale of intensities (perhaps, even "abstract" intensities, like probability or "hope") by referring to a scale in a completely different sensory system.
The END?

Not at all.
So far we talked mostly about the biological basis of mathematical thinking, something that can be called protomathematics.

(Aaron Sloman)
But mathematics starts with a child reflecting on

- what protomathematical structures of child’s mind tell the child;
- how this agrees with the physical world around;
- how this agrees with the cultural world around; and
- how this agrees with what adults tell the child.

In short,

**mathematics starts as meta-mathematics.**
Leo Harrington:

My stories involve no mathematics; but for me they involve some very primitive meta-mathematics, namely: **who is in control** of the meta-mathematics.

A dominant theme in dozens of stories:

**child’s fight for control** over mathematical concepts and structures crystallising in his/her mind.
Anna Borovik:

aged 9 was using the Russian word приручить, “to tame”, to describe accommodation of new concepts that she learnt at school: the concept had to become tame, obedient like a well trained dog.

The word was her secret, she never mentioned it to parents or teachers.
Social vs neurological: training of a inner dog
We have to do mathematics using the brain which evolved 30,000 years ago for survival in the African savanna.

Stanislas Dehaene, *The Number Sense*

Totally controlled mental operations: at most 16 bits per second

Visual processing in the brain: 10,000,000,000 bits per second.

Mathematics is a language for communication with subconsciousness.
Language of dog commands (“Fetch! Bite!”) is a social construct, a result of cultural evolution.
But a dog is an animal.
When two mathematicians meet, their inner dogs sniff each other.
Mathematics education is like lessons in dog training— with puppies left at home.
Heinrich Neuhaus 1888 – 1964

His book “The Art of Piano Playing” should be a compulsory reading for every teacher of mathematics.

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Heinrich Neuhaus, “The Art of Piano Playing”:

Learner’s progress is halted by accumulated neurological damage.

The same applies to mathematics.

Different methods of mathematics teaching damage different groups of children.
5. Mathematics in the Society
English banknote with a portrait of Adam Smith
The words on the note:
"The division of labour in pin manufacturing"
AN INQUIRY INTO THE
Nature and Causes
OF THE
WEALTH OF NATIONS.

By ADAM SMITH, LL.D. and F.R.S.
Formerly Professor of Moral Philosophy in the University of Glasgow.

IN TWO VOLUMES.
VOL. I.

LONDON:
PRINTED FOR W. STRAHAN; AND T. CADELL, IN THE STRAND.
MDCCCLXVI.
Book I, Chapter I: Of The Division of Labour

On pin manufacturing:

"One man draws out the wire; another straightens it; a third cuts it; a fourth points it; a fifth grinds it at the top for receiving the head; to make the head requires two or three distinct operations; to put it on is a peculiar business; to whiten the pins is another; it is even a trade by itself to put them into the paper; and the important business of making a pin is, in this manner, divided into about eighteen distinct operations."
Adam Smith’s conclusion:

Separation of the pin production process into 18 operations increases the productivity by factor of 240.
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The history of Western civilisation is the history of ever deepening division of labour.
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The consequences are profound.
Full proofs of mathematics built-in in a mobile phone or MP3 player are beyond understanding by most graduates from my departments.

(Without proofs, as a cookbook, it is taught to electronics engineering students)
In the emerging division of intellectual labour, mathematics is a 21st century equivalent of sharpening a pin.
In the emerging division of *intellectual labour*, mathematics is a 21st century equivalent of sharpening a pin.

Of course, the same is true about physics, chemistry, biology . . .

. . . although biology is perhaps not sharpening the pin but attaching a head, which, as Adam Smith remarks, in itself consists of two or three operations.
We have to admit that 95% of population do not need any mathematics beyond use of a calculator.

But what are the implications for mathematical education?
Collapse of the traditional pyramid of education
Rebranding mathematics

_The key to the success of our enterprise is the aggressive marketing of the religious product._
Rabbi of the Reformist Synagogue, Irvine, California, c. 1990

Why not rebrand mathematics as a tool of personal development and a spiritually enhancing activity?

Why not try to create an up-market brand of maths learning, for the top 5% who still need it?
Rebranding mathematics

Selective mathematical education will not work unless we know:

- What are mathematical abilities?
- What is the nature of mathematical intuition?
- What children actually do when they learn mathematics?
- What mathematicians actually do when they do mathematics?