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Where will the next generation of
UK mathematicians come from?
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A Dozen Problems

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Abstract. The paper contains a sample of archetypal mathematical problems, very elementary, but at the same time reflecting the principal paradigms of mathematical thinking. The aim of our exercise is to give a flavour of what we would expect from sixth form leavers who might be interested and able to pursue a professional career involving serious mathematics – not necessarily within the narrow domain of mathematics itself.

1. Introduction

It may seem optimistic, but we would like the Discussion Document which results from this meeting managed to bring out the importance of a number of key issues – some mathematical, some pedagogical, some bureaucratic, some political. This collection of problems is meant to provide food for thought on some of the mathematical and pedagogical issues.

One important issue is to assess the need for, and to begin to clarify what would constitute, a more appropriate curriculum and assessment framework – one which would lay a better mathematical foundation for large numbers ($\sim$ 40%?) of students, and which would provide appropriate extension work and targets for perhaps half this large group.

A second important issue is the need to re-educate those in the profession (including teachers and administrators) so that they become more aware of the kind of material which can be used over time to encourage, and to begin to identify, those with distinctively mathematical talents.

As a contribution to such ends we would like to try to develop a relatively short list of "archetypal mathematical problems", which are very elementary (in some sense), yet which capture on this elementary level some of the principal paradigms of genuinely "mathematical thinking". Such a collection of problems might help teachers and others appreciate the difference between mere "exercises", which all students need to practise on, and the kind of "problems" which more able students regularly need to struggle with.

This selection has been drafted as a preparatory document for the meeting to serve as a shared reference point (something for you to shoot at!), with a view to clarifying what we should perhaps expect from ($\sim$ 2000?) school leavers who have the kind of abilities to pursue a career involving serious mathematics – not necessarily within the narrow domain of mathematics itself. (In that sense, our list

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is the antithesis of Arnold's famous list of 100 mathematical problems which every mathematics or physics graduate should be able to solve [**1**].)

In a separate document, we shall attempt to outline some of the specific cognitive traits which contribute to mathematical abilities in children. Our aim is to press for simple changes which might lead to more school leavers who appreciate, and who can solve, problems of this kind, to attract them into mathematics departments and to ensure that there is a career structure which provides opportunities for those with the necessary talent to contribute to UK mathematics in the future.

Caveats. We feel that we have to make, however, some important caveats.

- These problems are **not** tests of mathematical abilities. If a boy or girl can solve any of them, he or she deserves some attention from the mathematics teacher. Nonetheless a child without previous experience of non-standard problems might still have distinctive mathematical abilities but fail to solve these particular problems.
- If a child is systematically exposed to non-standard problems, he or she quickly expands the range of problems accessible to him/her. Anyone who runs mathematical competitions probably knows children who could solve most of the problems on the list.
- To see how a child copes with a problem, his/her answer or even a detailed solution is not enough. You have to talk to the child while he/she works on a problem, and, without giving him/her any hints, trace his/her line of thought.
- The whole spirit of such a list is that it should be as "curriculum independent" as possible. However, in the absence of a well-designed school mathematics curriculum most potentially able students never develop the skills or attitudes which bring non-standard problems within range. Thus these problems should not be treated as strictly diagnostic. We would like to develop a collection with the property that:

  ($\star$) those who succeed on one or more of the problems given clearly have some genuine ability, and the greater their success, the greater their potential.

  However, widespread inability to engage with problems of the kind listed may indicate neither that the problems are inappropriate, nor that local mathematical potential has disappeared, but rather that the standard mathematical diet currently fails to lay the necessary foundations for sensible diagnosis to be entertained.
- The problems in this selection have different levels of "difficulty" and the sample is not as representative as one might wish. We welcome suggestions and criticisms.

2. Problems

PROBLEM 1. Tom takes two hours to do a job. Dick takes three hours to do the same job. Harry takes six hours for the job. How long would all three take working together?[1]

PROBLEM 2. It takes five days for a steamboat to get from St Louis to New Orleans, and seven days to return back from New Orleans to St Louis. How long will it take for a raft to drift from St Louis to New Orleans?

PROBLEM 3. You are told that some anglers caught some fish, but you are only given partial information. You know that one angler caught 20 fish (or more); that two anglers caught 19 fish (or more); and so on, with 19 anglers catching 2 fish or more, and 20 anglers catching at least 1 fish. If no-one caught more than 20 fish, how many fish did the anglers catch between them?

PROBLEM 4. Write all integers from 1 to 60 in a single row:

$$12345678910111213141516171819\ldots5960.$$

From this number, cross out 100 digits so that the remaining number is

(a) the smallest possible;
(b) the largest possible.

PROBLEM 5.

(a) The number

$$1000000000003000000000000700000000021$$

is the product of two smaller natural numbers. Find them.

(b) How many of the integers

$$11,\ 1001,\ 100001,\ 10000001,\ldots$$

are primes?

PROBLEM 6.

(a) "Our teacher Mr. Jones has more than 1000 books", said Tom.
"Oh no, he has less than 1000 books", said Gareth.
"Well, Mr. Jones definitely has at least one book", said Helen.
Only one of these statements is true; if so, how many books does Mr. Jones have?

(b) One hundred statements are written in the notebook:
This notebook contains 1 false statement.
This notebook contains 2 false statements.
This notebook contains 3 false statements.
...
This notebook contains 100 false statements.
Which of these statements is true?

PROBLEM 7. Here are several dates in Swahili:

[1] Tony Gardiner's Problem Solving Journal. The same problem is currently offered at the entry examination into the 9th form (that is, 15 year old students) of the Mathematics Correspondence School, Novosibirsk State University (in a slightly different wording, with lion, wolf and dog eating a sheep).

> tarehe tatu Disemba jumamosi; tarehe pili Aprili jumanne; tarehe nne Aprili jumanne; tarehe tano Octoba jumapili; tarehe tano Octoba jumatatu; tarehe tano Octoba jumatano.

The translations in English are given in random order:

> Monday 5 October; Tuesday 2 April; Wednesday 5 October; Sunday 5 October; Saturday 3 December; Tuesday 4 April.

Write in Swahili: Wednesday 3 April; Sunday 2 December; Monday 1 November.

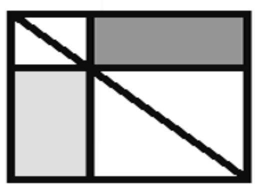

FIGURE 1. For Problem 8(a).

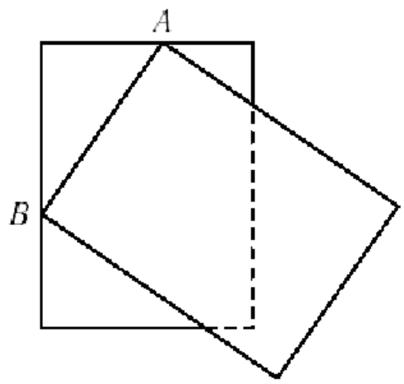

FIGURE 2. For Problem 8(b).

PROBLEM 8.
(a) In Figure 1, which of the two shaded rectangles is biggest?
(b) Two sheets of paper (of the same size) are placed one onto another as shown on Figure 2, so that corners A and B of the upper sheet lie on the sides of the bottom sheet, and one corner of the bottom sheet is covered. Which part of the bottom sheet is bigger: that covered by the upper sheet or the part uncovered? [**2**, Problem 163]

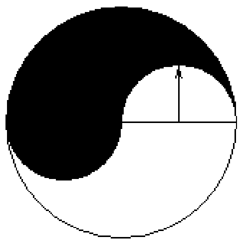

FIGURE 3. For Problem 9.

PROBLEM 9. The shaded region in Figure 3 is bounded by three semi-circles. Cut this region into four identical parts, i.e. parts of equal size and shape (but possibly of various orientation).[2]

PROBLEM 10. A bar of chocolate is subdivided by grooves into 40 segments, arranged in 5 rows and 8 columns. How many times does one have to break the bar to get all 40 segments?

PROBLEM 11. Estimate, approximately, how many solutions the following equation has:

$$x = 100 \sin x$$

PROBLEM 12. A rectangle of dimensions 19×91 is cut by straight lines parallel to its sides into $19 \times 91 = 1729$ equal squares of side 1. How many of these squares does a diagonal of the rectangle cross?[3]

3. Comments

Problem 1. Problems like that were a staple of arithmetic textbooks of yesteryear (well, of early 20th century). But though memories of pipes filling bathtubs may evoke negative vibes, the simple underlying methods for handling "rates" remain fundamental to an understanding of fractions and algebra, and are needed in many

[2]From the Introductory Assignment, Gelfand Correspondence Program in Mathematics, http://gcpm.rutgers.edu/problems.html.

[3]In an entry examination paper, Open Lyceum "All-Russian Correspondence Mathematics School", http://vzms.relline.ru/Math/join.shtml, this problem is given with bigger numbers: 199×991.

applications of mathematics. They have to be explicitly taught (and practised) if such problems are not to be completely inaccessible.

Problem 2. A non-standard repackaging of the same kind of ideas as in Problem 1. The first author (AB) met children who could immediately see the similarity: "Ah yes, it is about one over something and taking the sum, or difference, or whatever..."

Problem 3. One of possible solutions invokes a fundamental principle of Combinatorics: it is useful to count the same set in two different ways.[4]

Problem 4. A successful solver has to explicate the concepts "bigger" and "smaller" as applied to decimal numbers – only after that logical manipulations become possible.

Problem 5. Potentially able students need to be challenged to think flexibly and to make connections – such as the simple (but logically subtle and elusive) connection here between "factorisation" in algebra and in integer arithmetic. However, it is possible that a successful solver knows no algebra whatsoever – but in that case he or she should have a good ability to detect and explicate patterns in the behaviour of arithmetic procedures.

Problem 6. (a) One of the examples which shows how difficult it is to draw the line between the natural mathematical abilities and the "mathematical culture" absorbed at school. Besides an inclination to do combinatorial logic, the successful solver has to have a very clear understanding of the meaning of expressions "less" and "more". Also, the solution is not unique: it is interesting to see how children react to the problem being just a little bit undetermined.

(b) A trickier case of logic of self-referencing systems: statements refer to the number of other statements being valid.

Problem 7. Please notice that number 1 does not appear in the sample English translations. If you think that this makes it impossible to translate the last date, Monday 1 November, here is the answer: *tarehe mosi Novemba jumatatu.* Where does the numeral "one" sit in the Swahili sentences?

Problem 8. The two parts of this problem are loosely related.

(a) The first part represents the important class of problems in which the ingredients are entirely obvious, but the solver has to make the elusive move seeing that these obvious ingredients together with the principle "equals subtracted from equals are equal" cuts the knot. And much is gained if some at first use the more powerful and more general technique of employing algebra and ratios – only to meet the simpler idea later.

(b) It is interesting to compare the second part with one of the problems used in Celia Hoyles' and Dietmar Küchemann's study of development of the concept of proof in schoolchildren [**3**], see Figure 4.

As you can see from the comparison, Problem 8(b) is also about finding hidden symmetries, but in a less symmetrical setup.

Problem 9. One's first thought is likely to be that this must be impossible. So the problem tests both flexibility of thinking (since one has been challenged to succeed, so it is clear that one is missing something), and powers of visualisation. Actually, the shaded region can be divided in any number of identical parts.

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[4]By the way, it frequently happens that an elementary brainteaser contains a miniature version of a great mathematical theorem. In this problem, a mathematician will recognise the Fubini Theorem for Lebesgue integral.

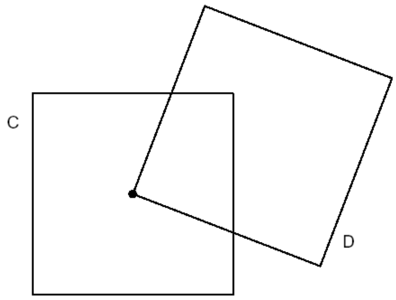

FIGURE 4. Squares C and D are identical. One corner of D is at the centre of C. What fraction of C is overlapped by D? Explain your answer.

Problem 10. The simple counting nature of the problem is intentionally obscured by excessive details.

Problem 11. One of the two problems on the list which formally overstep the pre-GCSE curriculum (the other one is Problem 5(b)). Of course, GCSE and A-level mathematics provides much more material for exciting problems of any level of difficulty.

Problem 12. This moves slightly beyond what English schoolchildren are currently taught. The solver needs to use that fact that 19 and 91 are relatively prime – yet "common factors" and "relative primeness" are treated rather superficially (perhaps in the context of fractions).

The problem is interesting for the dynamics of encapsulation/de-encapsulation: to solve the problem, you have to remember that numbers are results of counting processes (this also applies to Problem 10), and try to understand how a process fits into a static geometric picture.

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References

[1] V. Arnold, *Mathematical trivium*, Soviet Math. Uspekhi, 46 (1991), 225–232. (In Russian; English translations are easily available on the Internet.)

[2] Е. Г. Козлова, **Сказки и Подсказки**, Москва, МЦМНО, 2004. (ISBN 5-94057-142-5)

[3] D. Küchemann and C. Hoyles, *The quality of students' explanations on a non-standard geometry.* http://www.ioe.ac.uk/proof/cerme3.pdf

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