Mathematical Thinking
as
Experienced by a Mathematician

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Interdisciplinary Study of Mathematical Thinking
Nottingham, 21 November 2007
I. What is “mathematical thinking”? 

II. What is “mathematics”? 

III. Incidental observations on the mathematics/psychology boundary 

IV. “Invisible mathematical culture”
I. What is “mathematical thinking”?
Anonymous said…

I would caution everyone at this workshop not to confuse

“Mathematical Thinking”

with

“The thinking done by computer scientists and programmers”.

Unfortunately, most people who are not computer scientists believe these two modes of thinking to be the same.

The purposes, nature, frequency and levels of abstraction commonly used in programming are very different from those in mathematics.
Why is Computer Science different?

An example of a simple calculation with MATLAB.

```
>> t = 2

    t =

             2

>> 1/t

    ans =

             0.5000

(Floating point arithmetic for computations with rounding.)
```
Why is Computer Science different?

The same calculation with a different kind of integers:

```matlab
>> s=sym(’2’)

s =

2

>> 1/s

ans =

1/2
```

(“Symbolic integers”: coefficients in symbolic expressions.)
Why is Computer Science different?

Let us try to combine the two kinds of integers in a single calculation:

\[
\begin{align*}
\text{>> } & \text{s+t} \\
\text{ans } &= \\
& 4 \\
\end{align*}
\]

Is it a floating point number or a symbolic integer?
Why is Computer Science different?

Let us try to combine the two kinds of integers in a single calculation:

$$\frac{1}{s+t}$$

ans =

1/4

Hence the sum $s + t$ of floating point $t$ and symbolic $s$ is symbolic.
Why is Computer Science different?

Example above are written in C++.

In C++, even simplest mathematical objects and structures appear in form of (a potentially infinite variety of) classes linked by mechanisms of inheritance and polymorphism.

Indeed, computer scientists routinely use very high level of abstraction.
II. What is “mathematics”? 

Is there a boundary between “elementary” and “higher” mathematics?
What’s weird about 1, 11, 111, 1111 etc when you square them?

$1^2=1$, $11^2=121$. Keep on doing this with the other numbers. (If necessary use a calculator).

Solutions see page it counts up e.g. $111111^2 = 1234567654321$

But when you have $1,11,111,1111^2$ the answer is different. Figuring out (or using the calculator) what are the next square numbers in the pattern after $111111111^2$?

Solutions see page 123456790120987654321, 12345679012320987654321, 1234567901234320987654321 and 123456790123454320987654321!

Do you notice a pattern?

From a workbook of a schoolboy XY.
\[
1 \times 1 = 1 \\
11 \times 11 = 121 \\
111 \times 111 = 12321 \\
1111 \times 1111 = 1234321 \\
11111 \times 11111 = 123454321 \\
111111 \times 111111 = 12345654321 \\
1111111 \times 1111111 = 1234567654321 \\
11111111 \times 11111111 = 123456787654321 \\
111111111 \times 111111111 = 12345678987654321
\]

This is what XY meant.
But the pattern breaks at the next step and XY has already noticed that:

$$1111111111 \times 1111111111 = 1234567900987654321$$

The result is no longer symmetric. Why?
My writing on a whiteboard during my first meeting with XY.

I asked XY whether the symmetric pattern of results continued indefinitely.

XY instantly answered “No”.
To illustrate his point, XY’s wrote down, apparently from his memory, the first case when the pattern breaks.
“Good”–said I–but let us try to figure out why this is happening”, and wrote on the board:
“Yes”–said XY–“this is column multiplication”.

“And what are the sums of columns’?”

“1, 2, 3, 4, 3, 2, 1”–dictated XY to me, and I have written down the result:
**me:** “Will the symmetric pattern continue indefinitely?”

**XY:** “No – when there are 10 1’s in a column, 1 is added on the left and there is no symmetry.”

**me:** “Yes! Carries break the symmetry. But let us look at another example”–and I wrote:
XY was intrigued and made a couple of experiments:
It appeared from his behaviour that he was using mostly mental arithmetic, writing down the result, term by term, with pauses:

\[
\left(1+(1+x)^3\right) \left(1+(1+x)^2\right) = 1 + 4x + 6x^2 + 4x^3 + 3x^4 + 2x^5 + x^6
\]

XY said with obvious enthusiasm: “Yes, it is the same pattern!”
“Wonderful”—answered I—“let us see why this is happening. I’ll give you a hint: multiplication of polynomials can be written as column multiplication”, and started to write:
XY did not let me finish, grabbed the marker from my hand and insisted on doing it himself:

He stopped after he barely started the second line and said very firmly: “Yes, it is like with numbers”.
**me:** “Well – but will the pattern break down or will continue forever?”

[Pause...]

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me: “Well – but will the pattern break down or will continue forever?”

[Pause. . .]

XY: “No, it will not break down!”

me: “Why?”

XY: “Because when you add polynomials, the coefficients just add up, there are no carries.”
**me:** “You know, in mathematics polynomials are sometimes used to explain what is happening with numbers”.
me: “You know, in mathematics polynomials are sometimes used to explain what is happening with numbers”.

XY: “Yes, 10 is x.”
XY shows specific patterns of mathematical thinking.

For him, specialisation, inheritance, polymorphism are natural mechanisms of thinking: “10 is x”.

XY is prepared to use them at the very first encounter with a completely new problem.
XY and I share a common understanding that

**Even elementary arithmetic conceals sophisticated hidden structures**
Even elementary arithmetic conceals sophisticated hidden structures

Indeed, a carry, a digit that is transferred from one column of digits to another column of more significant digits during addition of two decimals, is defined by the rule

\[ c(a, b) = \begin{cases} 
1 & \text{if } a + b > 9 \\
0 & \text{otherwise}
\end{cases}. \]
Carry is a 2-cocycle from $\mathbb{Z}/10\mathbb{Z}$ to $10\mathbb{Z}$

\[ c : \mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z} \longrightarrow 10\mathbb{Z} \]

and is responsible for the extension of additive groups

\[ 0 \longrightarrow 10\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/10\mathbb{Z} \longrightarrow 0. \]

XY does not know cohomology (yet), but he is alert to potential presence of sophisticated hidden structures.
10-adic numbers:

- decimals infinite to the left:
  \[ \cdots 55554444333221, \]
  with usual addition and multiplication.

- Projective limit of
  \[ \cdots \to \mathbb{Z}/10^3\mathbb{Z} \to \mathbb{Z}/10^2\mathbb{Z} \to \mathbb{Z}/10\mathbb{Z} \to 0 \]

Exercise: Find zero divisors, explicitly.

For \( p \) prime, \( p \)-adics are better.
Arithmetical carry contains in itself a germ of a consistent and rigorous concept of limit.

**Euler:**

\[ 1 + 2 + 4 + 8 + 16 + \cdots + 2^n + \cdots = -1. \]

Does it make sense? Did it make sense for Euler?
**Gottfried Leibniz**: fully documented modern binary number system in the 17th century in *Explication de l'Arithétique Binaire*, 1703.

Leibniz’s system used 0 and 1, like the modern binary numeral system.

\[
\begin{align*}
0 &= 0 \\
1 &= 1 \\
2 &= 10 \\
3 &= 11 \\
4 &= 100 \\
5 &= 101 \\
6 &= 110 \\
7 &= 111 \\
&\vdots \\
2^n &= 10 \cdots 0 \quad (n \text{ zeroes})
\end{align*}
\]
TABLE

MEMOIRES DE L'ACADEMIE ROYALE

DES
NOMBRES

bres entiers au delà du double du
plus haut degré. Car icy, c'est com-
nue si on diroit, par exemple, que 111
ou 7 est la somme de quatre, de deux
& d'un.

Et que 1101 ou 15 est la somme de huit, quatre
& un. Cette propriété fera aux Efleyeurs pour
peifer toutes fortes de naiffes avec peu de poids,
& pourraient servir dans les monnoyes pour don-
ner plusieurs valeurs avec peu de pièces.

Cette expression des Nombres étant établie, fera à
faire tres facilement toutes fortes d'operations,

<table>
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<tr>
<th>Pour l'Addition</th>
<th>110</th>
<th>6</th>
<th>101</th>
<th>15</th>
<th>1101</th>
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<td>par exemple</td>
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<td>11</td>
<td>7</td>
<td>101</td>
<td>11</td>
<td>1000</td>
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| Pour la Sou-
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| Pour la Divi-
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Et toutes ces operations sont si ailes, qu'on n'a jamais
besoin de rien efilver ni deviner, comme il faut faire
dans la division ordinaire. On n'a point besoin non-plus
de rien apprendre par cœur icy, comme il faut faire dans
le calcul ordinaire, où il faut savoir, par exemple, que
6 & 7 font ensemble 13, & que 5 multiplié par 3
donne 15, suivant la Table d'aujuiour'hui en est, qu'on ap-
pelle Pythagorique. Mais icy tout cela se trouve & se
prouve de source, comme l'on voit dans les exemples pré-
cédens sous les lignes 3 & 6.
Back to Euler:

\[1 + 2 + 4 + 8 + \cdots = 1 + 10 + 100 + 1000 + \cdots = \cdots 111111111\]

\[\cdots 111111111 + 1 = \cdots 000000000\]

Hence in 2-adic arithmetic

\[1 + 2 + 4 + 8 + \cdots = -1\]
III. Incidental observations on the mathematics/psychology boundary:

Children and psychometric scales
Study of “able” kids reveals more about mathematical thinking than any number of observations of adults.
In XY, speed and fluency of generalisation suggests the ability of quickly form and operate with

scales of plausibilities of abstract statements.

Let us discuss scales.
“There is a tension in the field of neurology between the ‘single case study’ approach, an intensive study of just one or two patients with a syndrome, and . . . statistical analysis.”

“Most of the syndromes in neurology that have stood the test of time . . . were initially discovered by a careful study of single cases and I don’t know of even one that was discovered by averaging results from a big sample.”

Vilayanur Ramachandran
Systematic teaching of mathematics to very young children:
Vladimir Radzivilovsky (Russia, now Israel)

\[
\alpha \zeta \cos \sin' x = \frac{1}{\sqrt{1 - x^2}}
\]
\[
\sin(\alpha \zeta \cos \sin x) = x
\]
\[
\sin'(\alpha \zeta \cos \sin x) \cdot \alpha \zeta \cos \sin' x = 1
\]
\[
\cos(\alpha \zeta \cos \sin x) \cdot \alpha \zeta \cos \sin' x = 1
\]
\[
\sqrt{1 - x^2} \cdot \alpha \zeta \cos \sin' x = 1
\]
\[
\alpha \zeta \cos \sin' x = -\frac{1}{\sqrt{1 - x^2}}
\]

From a workbook of one of Radzivilovsky’s students: differentiation of inverse trigonometric functions.
And the student herself. Her name is Avital. She is 6.
Characteristic feature of Radzivilovsky’s method: direct appeal to psychophysiological substrate of mathematical intuition.

- Asking (very) young children to guess the weight and then weigh (in grams) every household item which they could fit on the scales.
- Asking children to estimate temperature (by touching water, say), and compare their feelings with a measurement by a thermometer.
- ...
A. Zvonkin, *Mathematics for little ones.*
Zvonkin's students.
And again we encounter scales.

A thermometer for measuring the “hope” to get two cubes of the same colour out of a box.
Two arithmetics: exact (symbolic) and approximate

“Exact arithmetic puts emphasis on language-specific representations and relies on a left inferior frontal circuit also used for generating associations between words. Symbolic arithmetic is a cultural invention specific to humans, and its development depended on the progressive improvement of number notation systems. […]”

“Approximate arithmetic, in contrast, shows no dependence on language and relies primarily on a quantity representation implemented in visuo–spatial networks of the left and right parietal lobes.”

Stanislas Dehaene
Brian Butterworth:

- the traditional way of teaching over-exploits verbal counting
- “approximate” arithmetic of magnitude suffers.
- proposes not to teach arithmetic until the age of 11.

His ideas are rooted in experiments of Louis Benezet in the 1920s:

- Benezet did not formally teach children arithmetic.
- Instead, a great deal of estimation activities in a real life setting.
One of the psychophysiological pillars of mathematical intuition:

**Mechanisms of cross-modal matching of scales of intensities of sensory experiences.**
One of (many) psychophysiological pillars of mathematical intuition:

**Mechanisms of cross-modal matching of scales of intensities of sensory experiences.**

**Conjecture:** the same mechanism matches

scales of plausibilities of mathematical statements.
Steven’s Power Law revisited?

**Steven (1906-1973):** for each perceptual continuum, there was a distinctive value of the exponent relating numerical judgment to stimulus intensity.

It appears that the modern psychometric community is rather critical about the Power Law.

**Dzhafarov and Colonius:** *Fechnerian Scaling*, more sophisticated metric approximation of dissimilarity spaces.
My naive question:

What is the nature of mechanisms which allow us to handle one scale of intensities (perhaps, even "abstract" intensities, like probability or "hope") by referring to a scale in a completely different sensory system?
My naive proposal:

— as a first step towards understanding of the nature of mathematical intuition:

a systematic review of existing psychometric literature and data

— with the aim to figure out what does it say about mathematical thinking.
IV. “Invisible mathematical culture”

(and some race issues)
A child in Zimbabwe, 1980s, pushing a wire toy automobile.
A wire toy pedal vehicle.
A wire toy automobile.

Did anyone ever study hidden mathematical content of children’s games and toys?
• What is hidden mathematical content of modern computer games?

• Has it changed over last 25 years?

• How the concept of “optimal strategy” entered the popular culture and became self-evident?
A question arising from Marcus Giaquinto’s talk:
Who had drawn the first genealogical tree?
Who had drawn the first genealogical tree?
More in the forthcoming book

**Mathematics under the Microscope**

(American Mathematical Society, 2008)

Available for download at

http://www.maths.manchester.ac.uk/~avb/micromathematics/