

Alexandre V. Borovik

Mathematics under the Microscope

Corrigenda and Addenda

The book itself is available from

[http://www.ams.org/
bookstore-getitem/item=mbk-71](http://www.ams.org/bookstore-getitem/item=mbk-71)

November 26, 2012

American Mathematical Society

Мы люди бедные и по бедности своей
мелкоскопа не имеем,
а у нас так глаз пристрелямши.

Левша

Acknowledgements

A number of addenda in the text were stimulated by comments from Mikhail Katz and Michael Livshits.

2.1 The starting point: mirrors and reflections



Fig. 0.1. The ability to recognize oneself in the mirror is equated, in popular culture, to self-awareness of a sentient being. Photo by roseoftimoth-woods. Source: *Wikipedia Commons* and *Flickr*. Licensed under Creative Commons Attribution 2.0 License.

3.2 Number sense and grammar

Varley et al. [1] describe cases of adult patients with brain damage who lost ability to process correctly reversed and embedded sentences (such as “The man killed the lion”, “The lion killed the man”, and “The man who killed the lion was angry”), but retained ability to manipulate with brackets in arithmetic expression (such as $5 \times (6+2)$). Brannon [2] used this work to claim “the independence of language and mathematical reasoning”.

We have here a case of a terminological confusion: for, me, “mathematical reasoning” is not re-use of parsing rules for arithmetic expression learned in childhood; I argue that as far as parsing is concerned, “mathematical reasoning” means generation of *new* parsing patterns.

3.5 Parsing, continued: do brackets matter?

Limitations of software

My comments on software are echoed in Christof van Nimwegens paper *The paradox of the guided user: assistance can be counter-effective* [3]

A recurring issue in usability guidelines is the importance of minimizing “user memory load”, also referred to as computational offloading (Scaife & Rogers, 1996). This means that the working memory (WM) of a user is relieved so that a maximum of cognitive resources can be devoted to the task. To achieve this, certain information can be externalized to the problem solver and thus be of assistance. Already two decades ago, Norman (1988) proposed the idea that knowledge might be as much in the world as it is in the head. He pointed out that the information embedded in technological artifacts (such as computer interfaces) was as important to task achievement as the knowledge residing in the mind of the user. [...] Externalization is to make parts of the interface context-sensitive, e.g. by hiding or disabling functions that are not applicable at the moment. By doing so, the user is “taken by the hand” by limiting choices and providing feedback (Van Oostendorp & De Mul, 1999). Examples are wizards, help-options and

¹ R. A. Varley, N. J. C. Klessinger, C. A. J. Romanowski, and M. Siegal, Agrammatic but numerate. *Proc. Nat. Acad. Sci. USA* 102 (2005), 3519-3524.

² E. M. Brannon, The independence of language and mathematical reasoning. *Proc. Nat. Acad. Sci. USA* 102 no. 9 (2005), 3177-3178.

³ Christof van Nimwegen, *The paradox of the guided user: assistance can be counter-effective*, <http://igitur-archive.library.uu.nl/dissertations/2008-0401-200600/nimwegen.pdf>. Accessed 27 September 2010.

grayed-out menu-items that do not permit using them, thus offering a context-sensitive interface where only possible actions are displayed. For example, in MS-Word, one cannot select “paste” from the menu when nothing is copied first. “Paste” shown in gray color indicates both the presence and the unavailability of the command. “Graying out” externalizes the applicability of an operator at a certain moment, and makes memorizing this information unnecessary.

In the opposite situation, when no such features are provided, a user has to internalize the information by himself, and store this information in his memory. The above definition of usability implies that systems are tools we use to accomplish a certain goal (text processing, spreadsheets, editing video, looking for information on the web). Rubin (1994) was probably right, though it can be interesting to look at what these presumed friendly systems actually do to humans and their behavior on a deeper level, and whether the highest achievable usability possible in the classical sense of the word is in fact always desirable.

Using experimental data, van Nimwegen concludes that “externalisation” harms learning. In his words,

In modern GUIs, users often click around without control, clicking and clicking on and on has become our second nature, perhaps partly due to the World Wide Web. The undo facilities that became almost mandatory in systems are mostly good and handy features, but perhaps they changed our ideas about the consequences of our actions in computer environments.

[...] one has to delve into human nature, and be aware of the mechanisms that are natural to us, which include, unfortunately, laziness and shallow behavior. There is nothing we like better than the feeling that things are being done for us, and in many cases, this can be exactly what is desired. However, in some situations one has to be careful to neglect these characteristics. This research concerns exploring the conditions under which motivation and deep contemplation for the task can be provoked to achieve better task performance, fewer errors, and having users that constantly are on top of the task.

Verbalization and vocalization

Verbalization and its stronger form—*vocalization*— is an important but increasingly neglected channel of interiorization of mathematics. School culture in various countries treats it differently; vocalization of arithmetic routines was a norm in Russian schools

of my time, but, apparently, not in England. This is a testimony from Alison Price⁴:

I do not have memories of a particular content of mathematics but do have vivid memories of myself as an 8 year old being taught in a very formal teaching context. The teacher would show us how to do a calculation on the board and we were then expected to work through examples in our maths books in silence.

My problem was that I could not learn new mathematics without being able to talk about what I was doing.

As a result I frequently was hit with a plastic ruler (this was allowed in the 1950s) for talking in mathematics class. As you can imagine this was not conducive to learning.

And this is from current Russian mathematics curriculum for ages 6–10:

The process of learning mathematics involves familiarisation with mathematical language and formation of speech skills: children learn to express propositions using mathematical terms and concepts, identify and emphasise words (phrases, etc.) that help to understand their meaning, raise questions in the course of completion of an assignment, choose evidence of correctness or incorrectness of the action performed, justify the steps of the solution, etc.

4.1 Parables and fables

These continues the theme of the subitizing/counting threshold.

I discovered an article [⁵] about the special role of number 6 in the works of Daniil Kharmis, the famous Russian absurdist writer. Actually, there is a strong feeling that Kharmis was writing about the classical subitising/counting threshold. Two samples: one in English translation, another is in original Russian (I cannot find a decent translation of the most mathematical of all Kharmis works).

FALLING-OUT OLD WOMEN

A certain old woman fell out of a window because she was too curious. She fell and broke into pieces.

Another old woman leaned her head out the window and looked at the one that had broken into pieces, but because she was too curious, she too fell out of the window fell and broke into pieces.

⁴ The Reverend Dr Alison J Price is English, female, Principal Lecturer in Mathematics and Religious Education in a British university.

⁵ А. Герасимова, Н. Каррик. К вопросу о значении чисел у Хармса: Шесть как естественный предел. (In Russian.) <http://xarms.lipetsk.ru/texts/gerasl.html>.

Then a third old woman fell out of the window, then a fourth, then a fifth.

When the sixth old woman fell out, I became fed up with watching them and went to Maltsevsky Market, where, they say, a certain blind man was presented with a knit shawl.

And another one, from the text “Я вам хочу рассказать . . .”:

заметим, что:

1. Две руки, две ноги, посередке сапоги.
2. Уши обладают тем же чем и глаза.
3. Бегать – глагол из-под ног.
4. Шупать – глагол из-под рук.
5. Усы могут быть только у сына.
6. Затылком нельзя рассмотреть что висит на стене.
17. Обратите внимание что после шестерки идет семнадцать. [6]

A translation of the last line: 17. Notice the six is followed by seventeen.

And, of course, I have to append Kharms’ glorious *A sonnet*⁷:

An amazing thing happened to me today, I suddenly forgot what comes first⁷ or 8.

I went to my neighbors and asked them about their opinion on this matter. Great was their and my amazement, when they suddenly discovered, that they couldnt recall the counting order. They remembered 1, 2, 3, 4, 5 and 6, but forgot what comes next.

We all went to a commercial grocery store, the one thats on the corner of Znamenskaya and Basseinaya streets to consult a cashier on our predicament. The cashier gave us a sad smile, took a small hammer out of her mouth, and moving her nose slightly back and forth, she said: - In my opinion, a seven comes after an eight, only if an eight comes after a seven.

We thanked the cashier and ran cheerfully out of the store. But there, thinking carefully about cashiers words, we got sad again because her words were void of any meaning.

What were we supposed to do? We went to the Summer Garden and started counting trees. But reaching a six in count, we stopped and started arguing: In the opinion of some, a 7 went next; but in opinion of others an 8 did.

We were arguing for a long time, when by some sheer luck, a child fell off a bench and broke both of his jaws. That distracted us from our argument.

And then we all went home.

⁶ Does anyone know a decent English translation of “Я вам хочу рассказать . . .”?

⁷ With thanks to Donald A. Preece who brought my attention to it.

And now I wish to mention an even more direct link between Kharms’ “A sonnet” and modern cognitive science.

Indeed notice that Kharms writes, in a poetic form, about what now is known as “mathematics anxiety”, and his observations find a direct confirmation in the modern cognitive science [8]:

Individuals with mathematics anxiety have been found to differ from their non-anxious peers on measures of higher-level mathematical processes, but not simple arithmetic. The current paper examines differences between mathematics anxious and non-mathematics anxious individuals in more basic numerical processing using a visual enumeration task. This task allows for the assessment of two systems of basic number processing: subitizing and counting. *Mathematics anxious individuals, relative to non-mathematics anxious individuals, showed a deficit in the counting but not in the subitizing range.* Furthermore, working memory was found to mediate this group difference. These findings demonstrate that the problems associated with mathematics anxiety exist at a level more basic than would be predicted from the extant literature.

More papers on subitizing: [9], [10].

4.5 Convexity and sensorimotor intuition

Olson [11] quotes Abranavel [12] to say that

young children made accurate diameter estimates well before they made accurate length estimates, both of which involve Euclidean concepts. This observation was traced back to the fact that young children automatically grasped the diameter by picking up the object, while length could not be apprehended without active search.

-
- ⁸ E. A. Maloney, E. F. Risko, D. Ansari, and J. Fugelsang, Mathematics anxiety affects counting but not subitizing during visual enumeration, *Cognition* 114, no. 2 (February 2010), 293–297. DOI:10.1016/j.cognition.2009.09.013.
- ⁹ M. LeCorre and S. Carey, One, two, three, four, nothing more: An investigation of the conceptual sources of the verbal counting principles. *Cognition* 105 (2007), 395–438.
- ¹⁰ S. K. Revkin, M. I. Piazza, L. Cohen, and S. Dehaene, Does subitizing reflect numerical estimation? *Psychological Science* 19 (2008), 607–614.
- ¹¹ D. R. Olson, **Cognitive Development: The Child’s Acquisition of Diagonality**. Lawrence Erlbaum Associates Inc., 1996. ISBN-10: 0805823026. ISBN-13: 978-0805823028.
- ¹² E. Abranavel, The development of intersensory patterning with regard to selected spatial dimensions. *Monographs of the Society for Research in Child Development* 33, 2 whole no. 118 (1968).

A classical study of tactile perception in young children is Piaget and Inhelder [13].

5.5 The vanishing point

Here is an example of a wonderfully consistent theological perspective on time:

In his *Systematic Theology*, Vol. III, Wolfhart Pannenberg argues that God as eternal comprehends the different moments of time simultaneously and orders them to constitute a whole or totality. The author contends that this approach to time and eternity might solve the logical tension between the classical notion of divine sovereignty and the common sense belief in creaturely spontaneity/ human freedom. For, if the existence of the events constituting a temporal sequence is primarily due to the spontaneous decisions of creatures, and if their being ordered into a totality or meaningful whole is primarily due to the superordinate activity of God, then both God and creatures play indispensable but nevertheless distinct roles in the cosmic process. [14]

6.2.1 Balls, bins and the Axiom of Extensionality

Thinking for themselves about potential and actual infinity, children can pick very subtle points. This is a one of childhood stories, from Elizabeth Frenkel [15]

The first that comes to my mind is the *Infinity* Hotel. I think I was 11 or 12 years old at that time. It was not that simple even for a finite number of guests. I remember talking about the problem with one of my parents—they were mathematicians by education. It appeared very strange to me that when we want to accommodate just two guests we cannot send them “to infinity”. I probed to settle them “at infinity”. Indeed there is always one more room! But everyone thought that this answer was not efficient. My parents (and a book) asked to name a concrete room number. What should be room numbers for keys that a porter has to handle to the guests? Such a “non-symmetry” of infinity appeared to me to be very unrealistic.

¹³ J. Piaget and B. Inhelder, **The Child’s Perception of Space**. Routledge and Kegan Paul, 1956.

¹⁴ J. A. Bracken, A new look at time and eternity, *Theology and Science*, 2 no. 1 (2004), 77-88. DOI: 10.1080/1474670042000196621.

¹⁵ EF is female, Russian, has a PhD in mathematics. This episode happened when she was 11 or 12 years old.

It is dangerous to invoke irrelevant arguments when talking to a bright child. Look how Lizzy turned the table on her parents' argument:

The case when an infinite number of guests suddenly arrived was perceived by me as more natural, even with the trouble of adding the infinity *at the end*. *But!* The issue of efficiency came to the fore here. How could this happen? For accommodating either finite or infinite number of guests we have to do about the same number of operations!

Lizzy was also unhappy with the Principle of Excluded Middle (at least when it is used in conjunction with the Axiom of Extensionality):

Russell's Paradox (of course, in the form of Barber's Paradox) caused resentment in me. Rephrasing my memories in today's language, my feelings can be described as follows. By definition, we have excluded the barber from the set of men who are shaved by him as well as from the set of men who are not shaved by him. Why then we ask which set the barber belongs to? Of course, my question did not prove that I had understanding of foundations of set theory, or even of any understanding of the essence of the problem, but still, perhaps it contained something . . .

6.2.5 Finite differences and asymptotic at zero

Vladimir Arnold's thesis that it is wrong to think of finite differences as approximations of differential operators is supported, with a brimstone-and-fire passion, by Doron Zeilberger; see his paper with the telling title "*Real*" analysis is a degenerate case of discrete analysis [16].

¹⁶ D. Zeilberger, "Real" analysis is a degenerate case of discrete analysis, in **New Progress in Difference Equations (Proc. ICDEA 2001)** (B. Aulbach, S. Elaydi, and G. Ladas, eds.). CRC Press, 2004. ISBN-10: 0415316758. ISBN-13: 978-0415316750. Available from <http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/real.html>.

6.3 Proofs by handwaving

A childhood testimony from AN which works as a comment on Lakoff and Núñez [¹⁷] and Núñez [¹⁸]

As for the limits, we were told to imagine an animal getting closer to point on the plane but not really arriving at it. This is a very poor description and gave me many years of unnecessary pain while dealing with limits. The thing is this animal might actually decompose itself into many different animals who have nothing to do with the first one and whoever acts more dramatically close to a certain point will guide the limit close to that point.

7.9 Winning ways

I quoted G. H. Hardy saying that the *solution* of a chess problem is

just that ‘proof by enumeration of cases ... which a real mathematician tends to despise.

This feeling of contempt spread much further than chess. I met quite a number of mathematicians who, say, hated traditional logic puzzles—and I happily share this attitude. As one of my colleagues put it:

I think for many mathematicians, perhaps even a majority, the appeal of the subject lies in being able to solve particular, special problems, such as logic puzzles, and being able to use (and know) special tricks for doing so. Those are the sort of mathematicians who excel at Olympiads and enjoy mathematical games.

For others of us, however, all that emphasis on puzzles and tricks was - and is - a complete waste of time. I always thought those puzzles were about as interesting as crossword puzzles or card games, activities which I always found “mind numbing and soul destroying”, to use the words of Yevtushenko. How is the world a better place after solving a cross-word puzzle? What have you, the solver, gained from solving a logic puzzle? Absolutely nothing, as far as I could tell.

¹⁷ G. Lakoff and R. Núñez, **Where Mathematics Comes From: How the Embodied Mind Brings Mathematics Into Being**. Basic Books, New York, 2000. **Where Mathematics Comes From: How the Embodied Mind Brings Mathematics Into Being**. Basic Books, New York, 2000.

¹⁸ R. Núñez, Do *real* numbers really move? Language, thought and gesture: the embodied cognitive foundations of mathematics, in **18 Unconventional Essays on the Nature of Mathematics** (R. Hersh, ed.). Springer, 2005, pp. 160–181.

My mathematical interest was always in the large-scale formal, abstract systems and structures of the subject, not in particular puzzles or tricks. I remember being thrilled at about age 14 to learn that “ $y = f(x)$ ”, that two different concepts which my teachers had tossed around loosely for some time, the vertical axis “ y ” in Cartesian co-ordinates and a function of some variable “ x ” were in fact the same thing. At last these two disparate subjects were connected, and lots of different things made sense as part of the same over-arching picture. The same experience happened a year or so later when we were first taught the differential calculus, and I could finally see a general theory behind what we had been doing with all those discussions of tangents to particular functions.

Some mathematicians certainly do like examples and particular problems and clever tricks. Others of us, however, think top-down and not bottom-up, and so desire general frameworks and abstract structures. Any effective education system should recognize the existence of at least these two styles of thinking, and not assume that all students are bottom-up thinkers.

Perhaps the emphasis on puzzles and tricks is fine for some mathematicians—e.g., Paul Erdos seems to have been motivated by puzzles and eager to solve particular problems. However, it is not fine for others—Alexander Grothendieck comes to mind as someone interested in abstract frameworks rather than puzzle-solving. Perhaps the research discipline of pure mathematics needs people of both types. If so, this even more reason not to eliminate all the top-down thinkers by teaching only using puzzles at school.

7.10 A dozen problems

Problem 10. Alexander Bogomolny offered a very simple solution:

Think of counting stones in a pile. This is done step-by-step, every time splitting a pile and announcing how many smaller piles are still there. The process stop when all piles consist of exactly one stone.

8.4 The triumph of the heuristic approach: Kolmogorov’s “5/3” law

One of the recent papers devoted to deduction of Kolmogorov’s cascade from the Navier-Stokes equations [¹⁹] contains a considerable bibliography.



Fig. 0.2. Woman teaching geometry. 1309–1316, France (Paris). The British Library. Source: *Wikimedia Commons*. Public domain.

From *Wikimedia*: Detail of a scene in the bowl of the letter ‘P’ with a woman with a set-square and dividers; using a compass to measure distances on a diagram. In her left hand she holds a square, an implement for testing or drawing right angles. She is watched by a group of students. In the Middle Ages, it is unusual to see women represented as teachers, in particular when the students appear to be monks. She may be the personification of Geometry. Illustration at the beginning of Euclid’s *Elementa*, in the translation attributed to Adelard of Bath.

8.6 Women in mathematics

A comment from TE:

¹⁹ C. Foias, M. S. Jolly, O. P. Manley, R. Rosa, and R. Temam, Kolmogorov theory via finite-time averages, *Physica D* 212 (2005), 245–270.

I have to comment on the “intrinsic competitiveness” of mathematics.

In mathematics olympiads and in professional mathematics, competitiveness is ritualized and contained by ethical rules. In my olympiad training session, you were expected to explain things to newer participants even though they might well be better than you at the competition. In olympiads and real life, I have found that this kind of ethics almost always corresponded to mathematical capability.

So here is my anecdote on competitiveness:

In primary school (age 6–10), I had about half the running speed of any other child. On the other hand, I understood everything in maths immediately and was very quick with mental calculations. In sports, we had weekly running competitions of groups by four, where my group never had a chance at all. In maths, we had weekly speed calculating competitions where each pair of students were posed a question and the quicker one would remain standing until one student was left and got a little prize. I won every time and was only allowed to participate every other week.

The lesson that good performance in math is bad for your social standing with your peers could not have been more evident. [²⁰]

A recent massive survey of research in gender issues in mathematics has appeared in [²¹].

Competitiveness and aggression

This is an important theme in the discussion of the place and role of women in mathematics: an extreme competitiveness and aggression in research mathematics. This is a testimony from my colleague:

Part of my anger and frustration at school was that so much of this subject that I loved, mathematics, was wasted on what I thought was frivolous or immoral applications: frivolous because of all those unrealistic puzzles, and immoral because of the emphasis on competition (Olympiads, chess, card games, gambling, etc). I had (and retain) a profound dislike of competition, and I don’t see why one always had to demonstrate one’s abilities by beating other people, rather than by collaborating with them. I believed that “playing music together”, rather than “playing sport against one another”, was a better metaphor for what I wanted to do in life, and as a mathematician.

²⁰ TE is female, learned mathematics in German. She has a PhD in Mathematics (and a Gold Medal of an International Mathematical Olympiad), teaches mathematics at an university.

²¹ D. F. Halpern, C. P. Benbow, D. C. Geary, R. C. Gur, J. S. Hyde, and M. A. Gernbacher, The science of sex differences in science and mathematics, *Psychological Science in the Public Interest* 8 (2007), 1–51.

Indeed, the macho competitiveness of much of pure mathematics struck me very strongly when I was an undergraduate student: I switched then to mathematical statistics because the teachers and students in that discipline were much less competitive towards one another. For a long time, I thought I was alone in this view, but I have since heard the same story from other people, including some prominent mathematicians. I know one famous category theorist who switched from analysis as a graduate student because the people there were too competitive, while the category theory people were more co-operative.

It may be worth mentioning that I am male. In other words, a dislike of competitiveness is not confined to women. The statistics department I entered as an undergraduate, for example, had no women in it, yet was much less competitive than the pure mathematics department (which had once been headed by a woman). I think it is disciplinary tradition rather than gender that is the key factor here.

I am now a Computer Scientist. I have also found differences in the competitiveness of people in different sub-domains of CS. To generalize greatly, I have found people in Artificial Intelligence (AI) much less macho and competitive than those in (say) Algorithm and Complexity Theory. Within AI, people in (say) Argumentation are generally much less macho and competitive than those in Game Theory and Mechanism Design. In each case, the more formal and mathematical the domain, the more competitive it tends to be. It could be that these domains have acquired their cultures from mathematicians, while the other domains have been less influenced by the culture of mathematics.

Competitiveness as a virtue

A completely different approach to competitiveness comes, interestingly, from the Russian tradition of education in mathematics and physics.

I quote a few fragments from an article *Physics and Mathematics School in a third of century* written in 1988 about my *alma mater*, the preparatory boarding school of Novosibirsk State University in Siberia, by my physics lecturer at the School, Evgenii Ivanovich Bichenkov, and translated in English by my old friend Owl (*Otus Persapience*), also a former student of the School.

The document that Owl and I are to offer to your attention is a manifesto of *meritocratic eliticism* in education, a recipe for building a highly selective²² and academically intensive school. A word of warning: Bichenkov's paper was written in a different historic

²² The catchment area of the School covered huge tracts of Siberia, Russian Far East and former Soviet Central Asia and had population of at least 20 million people.

epoch, in different socio-economic, political and cultural environments. It is not a recipe for modern day Britain!

What follows are quotes from Bichenkov; he is obviously very fond of competition.

[...] What new was brought by the school into the practice of school education? What are the principal results of its activities in teaching basics of sciences at a school level? [...]

So, what has been achieved by selection of students? I am deeply convinced that the very fact of selection and creation of childrens collective on the basis of selection is beneficial for a child. When they come from their schools, where all roles and places have been already assigned and fixed, children start their internal competitions for distribution over the scale of their hierarchy of values. They cannot not to do that—such is their nature and their age. It is important that at that age they are offered decent moral and human rules of competitions and shown some good examples. It appears that the Novosibirsk FMSH has succeeded in that.

Next. To what degree selection was determined by true abilities? Did results match the declared aims?

Here I cannot give a definite answer. In many ways, the selection can still be affected by chance. The selection is obviously influenced by personal aspirations and interests of the child, by the family, teachers, friends, acquaintances; the results of olympiads are affected by competitiveness, persistence, level of maturity, after all. And of course, choice manifests the personality of the teacher, examiner. [²³]

Please notice that Bichenkov accepts as inevitable that different examiners apply different criteria for selection; it does not matter for him whether the criteria are uniform; what matters is that they should be fair and allow the examiner to select the best candidates.

11.4 Mathematics and Origami

The mathematical history of Origami, in particular, the history of solving cubic equations by means of Origami, found a wonderful exposition in Thomas Hull's paper [²⁴], complete with a very detailed bibliography. Giorgio Ferrarese's website [²⁵] contains texts of a number of historic sources.

²³ A fuller text can be found in my blog post *Manifesto of meritocratic eliticism*, <http://micromath.wordpress.com/2010/09/28/manifesto-of-meritocratic-eliticism-2/>.

²⁴ T. C. Hull, Solving cubics with creases: The work of Beloch and Lill, *Amer. Math. Monthly* 118 (2011), 307–315. DOI:10.4169/amer.math.monthly.118.04.307.

²⁵ G. Ferrarese, <http://www2.dm.unito.it/paginepersonali/ferrarese/paperfolding/>. Accessed 5 May 2011.

