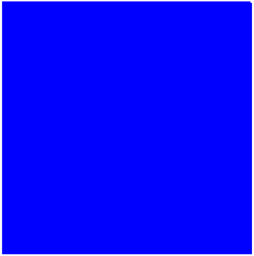
Nonlinear Solid Mechanics

Andrew Hazel

Introduction

- Typically, want to determine the response of a solid body to an applied load.
- If a solid body is not rigid, then it *can* deform.



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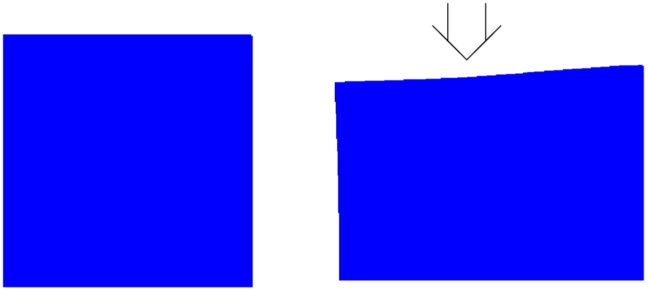

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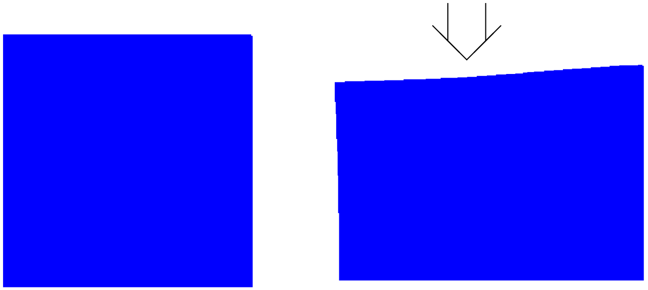
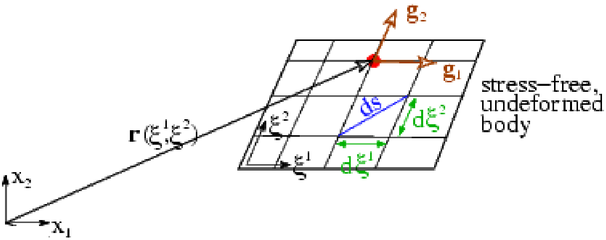


- **Question:** How do we measure the (finite) deformation of a deformable body?

Lagrangian coordinates

- We parametrise the position vector to any material point within the *undeformed* body, **r** by Lagrangian coordinates, ξ^i.

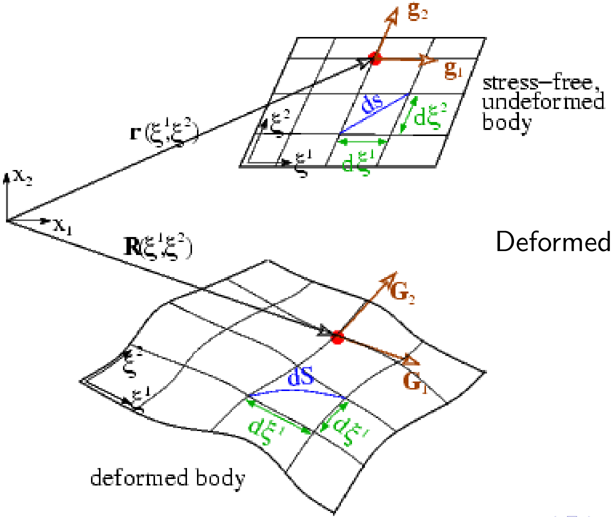


Tangent vector

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial \xi^i}$$

Lagrangian coordinates

- As the body deforms, the Lagrangian coordinates remain “attached” to the same material points in the body.



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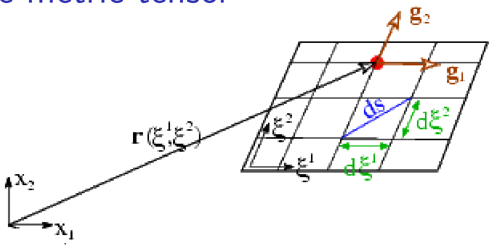
Deformed position, $\mathbf{R}(\xi^i)$

Tangent vector

$$\mathbf{G}_i = \frac{\partial \mathbf{R}}{\partial \xi^i}$$

The metric tensor



- A vector line segment $\mathbf{dr} = \mathbf{g}_i \, \mathrm{d}\xi^i$ has length (squared)

$$(\mathrm{d}s)^2 = \mathbf{dr}\cdot\mathbf{dr} = \mathbf{g}_i \, \mathrm{d}\xi^i \cdot \mathbf{g}_j \, \mathrm{d}\xi^j = g_{ij} \, \mathrm{d}\xi^i \, \mathrm{d}\xi^j,$$

where

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j,$$

is called the (covariant) metric tensor.

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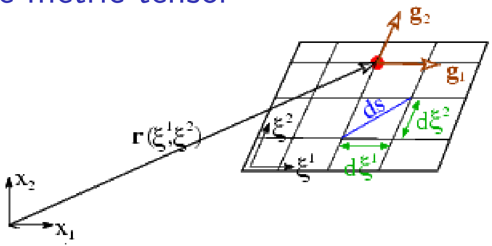

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  where

  $$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j,$$

  is called the (covariant) metric tensor.
- g_{ij} expresses information about the length of material lines in the undeformed body.
- If the global and Lagrangian coordinates coincide $g_{ij} = \delta_{ij}$.

The strain tensor

- An objective measure of the deformation (strain) is given by the Green strain tensor

$$\gamma_{ij} = \frac{1}{2}\left(G_{ij} - g_{ij}\right).$$

- g_{ij} is the metric tensor of the undeformed configuration.
- G_{ij} is the metric tensor of the deformed configuration.

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- Decompose the deformed position into

$$\mathbf{R}(\xi^i) = \mathbf{r}(\xi^i) + \epsilon \mathbf{u}(\xi^i),$$

where $\mathbf{u}(\xi^i)$ is the displacement field and $\epsilon \ll 1$, then

$$\mathbf{G}_i = \mathbf{g}_i + \epsilon \frac{\partial \mathbf{u}}{\partial \xi^i} \quad \Rightarrow \quad G_{ij} = g_{ij} + \epsilon \left(\mathbf{g}_i \cdot \frac{\partial \mathbf{u}}{\partial \xi^j} + \mathbf{g}_j \cdot \frac{\partial \mathbf{u}}{\partial \xi^i} \right) + O(\epsilon^2).$$

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- If ξ^i are chosen to be global Cartesian coordinates

$$\gamma_{ij} \approx \epsilon\,\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad \text{the infinitessimal strain tensor.}$$

Example

- Consider the simple radial expansion of a unit cube $0 \leq \xi^i = x^i \leq 1$, with deformed position given by

$$\mathbf{R} = 2\,\mathbf{r} = 2\,\mathbf{x}.$$

- Lagrangian coordinates are global Cartesian coordinates so

$$\mathbf{g}_i = \frac{\partial\,\mathbf{x}}{\partial x^i} = \mathbf{e}_i \text{ (a unit vector)} \quad \Rightarrow \quad g_{ij} = \delta_{ij}.$$

- The deformed metric tensor is

$$\mathbf{G}_i = \frac{\partial\,2\mathbf{x}}{\partial x^i} = 2\mathbf{e}_i \quad \Rightarrow \quad G_{ij} = 4\delta_{ij}.$$

- Hence the strain tensor is

$$\gamma_{ij} = \frac{3}{2}\delta_{ij},$$

- Note that the infinitessimal strain tensor (with $\epsilon = 1$) would give $\gamma_{ij} \approx \delta_{ij}$

An aside: Non-Cartesian tensors

- Starting from the position vector as a function of the Lagrangian coordinates $\mathbf{r}(\xi^i)$, we found the tangent vectors

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial \xi^i}.$$

- For a general set of coordinates, ξ^i, these vectors are not necessarily orthonormal,

$$\mathbf{g}_i \cdot \mathbf{g}_j \neq \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

- We **define** another set of vectors $\mathbf{g}^j$ so that

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

- The "up-down" index notation is used to simplify notation.

An aside: Non-Cartesian tensors

- We decompose $\mathbf{r}$ into global Cartesian base vectors

$$\mathbf{r} = \sum_k r^k \mathbf{e}_k,$$

  where $\mathbf{e}_k$ is a unit vector in the k-th global Cartesian direction.
- Hence,

$$\mathbf{g}_i = \frac{\partial r^k}{\partial \xi^i} \mathbf{e}_k,$$

- It follows that

$$\mathbf{g}^j = \frac{\partial \xi^j}{\partial r^n} \mathbf{e}_n,$$

  ... from which we deduce that

$$\mathbf{g}_i = g_{ij}\, \mathbf{g}^j \quad \text{and} \quad \mathbf{g}^j = g^{ji}\, \mathbf{g}_i, \quad \text{where } g^{ji} = \mathbf{g}^j \cdot \mathbf{g}^i.$$

Forces and loads

- A deformable body is typically subject to surface loads (or tractions), $\mathbf{t}$, and body forces, $\mathbf{F}$.
- The stress vector $\mathbf{t}$ on a surface ΔS within the strained body is defined by

$$\mathbf{t} = \lim_{\Delta S \to 0} \frac{\boldsymbol{\Delta}\mathbf{F}}{\Delta S},$$

  where $\boldsymbol{\Delta}\mathbf{F}$ is the (statically equivalent) force acting on the surface.
- $\mathbf{t}$ represents the force per unit area exerted by the material located to one side of the surface on that to the other.

The stress tensor

- We consider a force balance on an infinitessimal tetrahedron in the deformed configuration.
- Three faces are aligned with the planes $\xi^i = \text{const}$ and spanned by the other two covariant (lower index) tangent vectors $\mathbf{G}_j$.
- The vector representation of the face $\xi^i = \text{const}$ is

$$\frac{\mathbf{G}^i \Delta S_i}{2\sqrt{G^{ii}}},$$

  where $\mathbf{G}^i/\sqrt{(G^{ii})}$ is a unit vector normal to the face and the area of the face is $\Delta S_i/2$.
- Note that we have had to use the contravariant (upper index) vector to ensure orthogonality.

The stress tensor

- Vector representation of the remaining face is $\mathbf{n}\Delta S/2$ and so

$$\mathbf{n}\Delta S = \sum_i \frac{\mathbf{G}^i \Delta S_i}{\sqrt{(G^{ii})}}.$$

- Decomposing the normal $\mathbf{n} = n_i \mathbf{G}^i$, then

$$n_i \sqrt{(G^{ii})} \Delta S = \Delta S_i.$$

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$$\mathbf{t}\Delta S = \sum_i \mathbf{t}_i \Delta S_i,$$

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The stress tensor

$$\mathbf{t} = \sum_i n_i \sqrt{(G^{ii})}\mathbf{t}_i,$$

- $\mathbf{t}$ is invariant if $\mathbf{n}$ remains the same (Cauchy's principle).
- However, n_i are components of a covariant vector so $\mathbf{t}_i\sqrt{(G^{ii})}$ must be contravariant.
- In other words
$$\mathbf{t}_i\sqrt{(G^{ii})} = \tau^{ij}\mathbf{G}_j.$$
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- The quantity τ^{ij} is called the stress tensor.
- Physical components of the stress tensor are obtained by expressing the stress vectors in terms of unit tangent vectors

$$\mathbf{t}_i = \sum_j \sigma_{(ij)}\mathbf{G}_j/\sqrt{(G_{jj})} \quad \Rightarrow \quad \sigma_{(ij)} = \sqrt{(G_{jj})/(G^{ii})}\,\tau^{ij}.$$

Rate of Work

- In the strained body, the total rate of work is

$$RW = \iint \mathbf{t} \cdot \dot{\mathbf{R}} \, \mathrm{d}S + \iiint \left(\mathbf{F} - \rho \ddot{\mathbf{R}} \right) \cdot \dot{\mathbf{R}} \, \mathrm{d}V,$$

where the $\dot{\mathbf{R}}$ is the velocity of the material and $\ddot{\mathbf{R}}$ is its the acceleration.

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$$\Rightarrow RW = \iint \mathbf{T}_i \cdot \dot{\mathbf{R}} \frac{n_i}{\sqrt{G}} \, \mathrm{d}S + \iiint \left(\mathbf{F} - \rho \ddot{\mathbf{R}} \right) \cdot \dot{\mathbf{R}} \, \mathrm{d}V,$$

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  where $\mathbf{T}_i = \mathbf{t}_i \sqrt{(G G^{ii})}$ and $G = \det G_{ij}$.
- Now use the divergence theorem (see Green & Zerna)

$$\iint a^i n_i \, \mathrm{d}S = \iiint \frac{1}{\sqrt{G}} \frac{\partial}{\partial \xi^i} (a^i \sqrt{G}) \, \mathrm{d}V.$$

Rate of Work

- The rate of work becomes

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$$= \iiint \frac{1}{\sqrt{G}} \mathbf{T}_i \cdot \frac{\partial \dot{\mathbf{R}}}{\partial \xi^i} + \frac{1}{\sqrt{G}} \left[\frac{\partial \mathbf{T}_i}{\partial \xi^i} + \sqrt{G} \left(\mathbf{F} - \rho \ddot{\mathbf{R}} \right) \right] \cdot \dot{\mathbf{R}} \, \mathrm{d}V,$$

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- But recall

$$\gamma_{ij} = \frac{1}{2}\left(G_{ij} - g_{ij}\right),$$

so

$$\dot{\gamma}_{ij} = \frac{1}{2}\frac{\partial}{\partial t}\left(\mathbf{G}_i\cdot\mathbf{G}_j\right) = \frac{1}{2}\left(\mathbf{G}_i\cdot\frac{\partial\dot{\mathbf{R}}}{\partial\xi^j} + \frac{\partial\dot{\mathbf{R}}}{\partial\xi^i}\cdot\mathbf{G}_j\right).$$

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- Hence

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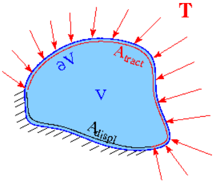
- τ^{ij} and γ_{ij} are a work conjugate pair.

The principle of virtual diplacements



- Consider a deformable body that is load by a surface traction **T** and a body force **F**
- The body is subject to a virtual displacement $\delta\mathbf{R}$

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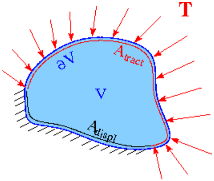

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- The virtual work induced by the displacement is

$$\iiint \left(\mathbf{F} - \rho\frac{\partial^2\mathbf{R}}{\partial t^2}\right) \cdot \delta\mathbf{R}\, \mathrm{d}V + \iint \mathbf{T} \cdot \delta\mathbf{R}\, \mathrm{d}S$$

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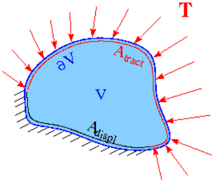

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which must be balanced by a change of internal energy

$$\iiint \tau^{ij}\delta\gamma_{ij}\,\mathrm{d}V$$

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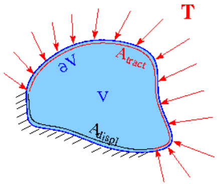

- Consider a deformable body that is load by a surface traction **T** and a body force **F**
- The governing variational principle is

$$\iiint \tau^{ij}\delta\gamma_{ij}\,\mathrm{d}V - \iiint \left(\mathbf{F} - \rho\frac{\partial^2\mathbf{R}}{\partial t^2}\right)\cdot\delta\mathbf{R}\,\mathrm{d}V - \iint \mathbf{T}\cdot\delta\mathbf{R}\,\mathrm{d}S = 0.$$

The principle of virtual displacements

- The variation of the strain is given by

$$\delta\gamma_{ij} = \frac{1}{2}\left(\mathbf{G}_i \cdot \delta\frac{\partial \mathbf{R}}{\partial \xi^j} + \delta\frac{\partial \mathbf{R}}{\partial \xi^i} \cdot \mathbf{G}_j\right).$$

- Using the symmetry of the stress tensor we can write the variational principle as

$$\iiint \tau^{ij} \frac{\partial \mathbf{R}}{\partial \xi^i} \cdot \delta\frac{\partial \mathbf{R}}{\partial \xi^j} - \left(\mathbf{F} - \rho\frac{\partial^2 \mathbf{R}}{\partial t^2}\right) \cdot \delta\mathbf{R}\, \mathrm{d}V - \iint \mathbf{T} \cdot \delta\mathbf{R}\, \mathrm{d}S = 0.$$

Working with the variational principle

- The integral is over the deformed domain, which is unknown.
- It is much more convenient to integrate over the undeformed domain.
- We already know the mapping from undeformed to deformed domain $\mathbf{R}(\xi^i)$.
- The Jacobian of the mapping is $\sqrt{G/g}$.
- Hence $\mathrm{d}V = \sqrt{G/g}\,\mathrm{d}V_0$ and

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$$\iiint \sigma^{ij} \frac{\partial \mathbf{R}}{\partial \xi^i} \cdot \delta \frac{\partial \mathbf{R}}{\partial \xi^j} - \left(\mathbf{f} - \rho_0 \frac{\partial^2 \mathbf{R}}{\partial t^2} \right) \cdot \delta \mathbf{R}\, \mathrm{d}V_0 - \iint \mathbf{T} \cdot \delta \mathbf{R}\, \mathrm{d}S = 0.$$

where $\sigma^{ij} = \sqrt{G/g}\,\tau^{ij}$ is the second Piola–Kirchhoff stress tensor,
$\mathbf{f}$ is the body force per unit undeformed volume,
ρ_0 is the undeformed density.

Finite element approximation of the Largange coordinates

- The undeformed position is given by

$$\mathbf{r}(\xi^i) = r^k(\xi^i)\mathbf{e}_k.$$

- If there are no special symmetries, choose Lagrangian coordinates as the global Cartesian coordinates.

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- Undeformed metric tensor is Kronecker delta $g_{ij} = \delta_{ij}$.
- Approximate the Lagrangian coordinates by a finite element basis

$$\xi^i = \sum_I \hat{\xi}^i_I \psi_I,$$

$\hat{\xi}^i_I$ is the i-th Lagrangian coordinate at the I-th node.

Finite element approximation of the variational principle

- Use the same basis function for the unkown positions (isoparametric mapping)

$$R^k = \sum_I \hat{R}_I^k \psi_I.$$

- The basis functions are fixed so

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Finite element approximation of the variational principle

- Use the same basis function for the unkown positions (isoparametric mapping)

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- Thus the principle of virtual displacements becomes

$$\sum_I \iiint \left[\sigma^{ij} \frac{\partial R^k}{\partial \xi^i} \frac{\partial \psi_I}{\partial \xi^j} - \left(f^k - \rho \frac{\partial^2 R^k}{\partial t^2} \right) \psi_I \right] \delta \hat{R}_I^k \, \mathrm{d}\xi^1 \, \mathrm{d}\xi^2 \, \mathrm{d}\xi^3$$

$$- \iint T_k \psi_I \delta \hat{R}_I^k \, \mathrm{d}S = 0.$$

Finite element approximation of the variational principle

- The discrete variations may be taken outside the integrals

$$\sum_I \left\{ \iiint \left[\sigma^{ij} \frac{\partial R^k}{\partial \xi^i} \frac{\partial \psi_I}{\partial \xi^j} - \left(f^k - \rho_0 \frac{\partial^2 R^k}{\partial t^2} \right) \psi_I \right] \mathrm{d}\xi^1 \, \mathrm{d}\xi^2 \, \mathrm{d}\xi^3 \right.$$

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- The variations of the nodes are independent, so the terms in braces give one discrete equation for each nodal unknown.

$$\iiint \left[\sigma^{ij} \frac{\partial R^k}{\partial \xi^i} \frac{\partial \psi_I}{\partial \xi^j} - \left(f^k - \rho_0 \frac{\partial^2 R^k}{\partial t^2} \right) \psi_I \right] \mathrm{d}\xi^1 \, \mathrm{d}\xi^2 \, \mathrm{d}\xi^3$$

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- These may be assembled in an element-by-element manner.

Summary of the method

- Divide the undeformed domain into elements.
- For each element compute the contribution to the discrete volume residual

$$\mathcal{R}_{kl} = \iiint \left[\sigma^{ij} \frac{\partial R^k}{\partial \xi^i} \frac{\partial \psi_l}{\partial \xi^j} - \left(f^k - \rho_0 \frac{\partial^2 R^k}{\partial t^2} \right) \psi_l \right] \mathrm{d}\xi^1 \, \mathrm{d}\xi^2 \, \mathrm{d}\xi^3$$

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$$\iint \left[T_k \psi_l \right] \mathrm{d}S.$$

- Note that this integral is over the **deformed** surface
- Assemble the contributions into a global residuals vector.
- Compute the Jacobian (by finite differences if necessary).
- Solve the linear system.

Constitutive Laws

- Assembling the residuals requires knowledge of the stress tensor.
- For an elastic material, the stress depends only on the current state of strain
$$\sigma^{ij}(\gamma_{jk}).$$
- The specific relationship between stress and strain is known as a constitutive law.
- Given a constitutive law and a compressible material then we simply compute
$$\sigma^{ij}\left(\frac{1}{2}(G_{ij}-g_{ij})\right),$$
at all the integration points within the element.
- What about incompressible materials?

Incompressible Solid Mechanics

- If a solid material is incompressible, its volume cannot change

$$\det G_{ij} = \det g_{ij} \tag{1}$$

- Enforce the condition (1) by a Lagrange multiplier that plays the role of a pressure so that

$$\sigma^{ij} = -p\, G^{ij} + \bar{\sigma}^{ij}(\gamma_{kl}),$$

where $\bar{\sigma}^{ij}$ is the deviatoric part of the stress tensor.

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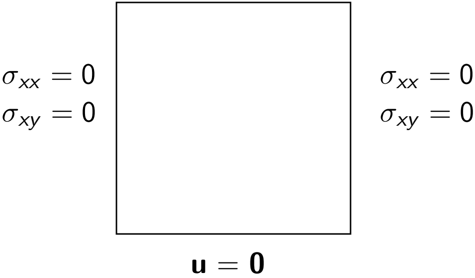
- If material is "nearly" incompressible it is also advantageous to use a mixed formulation.

Example problem: Compression of a block

- A square block of material is compressed.
- What are appropriate boundary conditions?



- Fix the base vertically (and horizontally?)
- Leave the sides traction free (do nothing).

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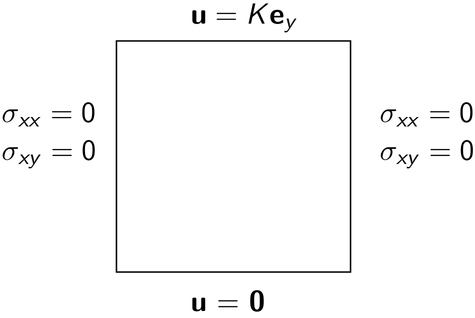

- Fix the base vertically (and horizontally?)
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- Impose a displacement on the top.

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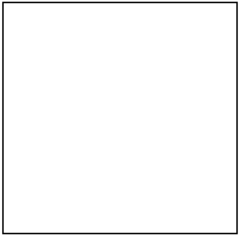
$$\sigma_{ij} n_j = -p n_i$$

$$\sigma_{xx} = 0 \qquad \sigma_{xx} = 0$$

$$\sigma_{xy} = 0 \qquad \sigma_{xy} = 0$$



$$\mathbf{u} = \mathbf{0}$$

- Fix the base vertically (and horizontally?)
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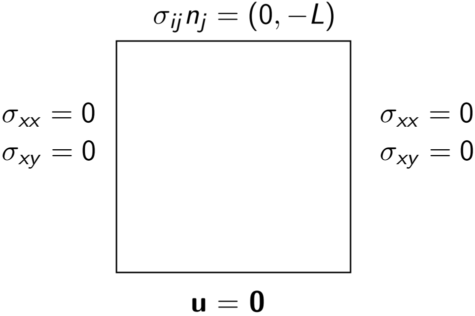

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- Impose a vertical load on the top.

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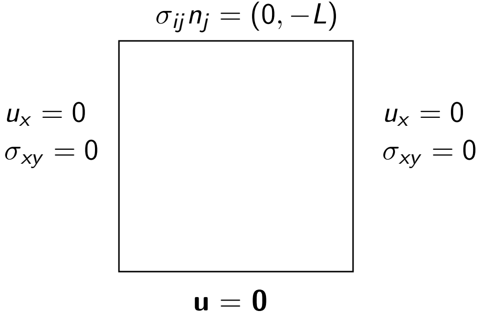
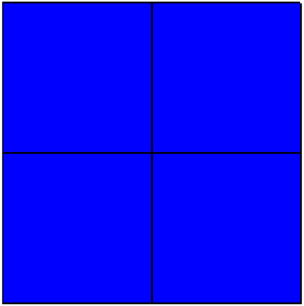


- Fix the base vertically (and horizontally?)
- Constrain sides horizontally
- Impose a vertical load on the top.
- Incompressible material?

Example problem: Compression of block

- Incompressible Mooney-Rivlin material
- Loaded on top by $P\cos(x)\mathbf{N}$



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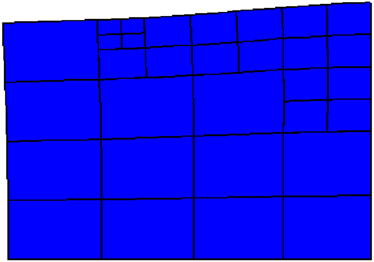

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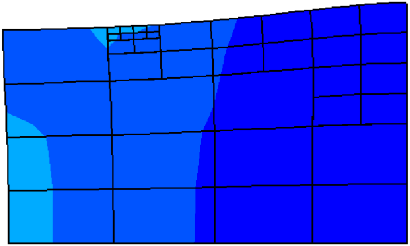

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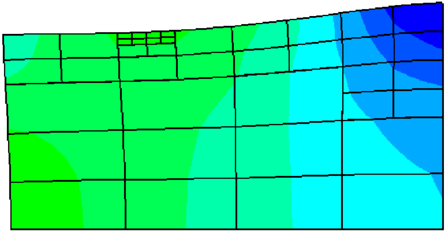

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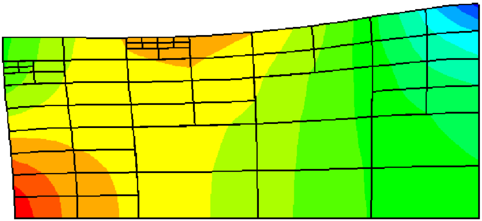

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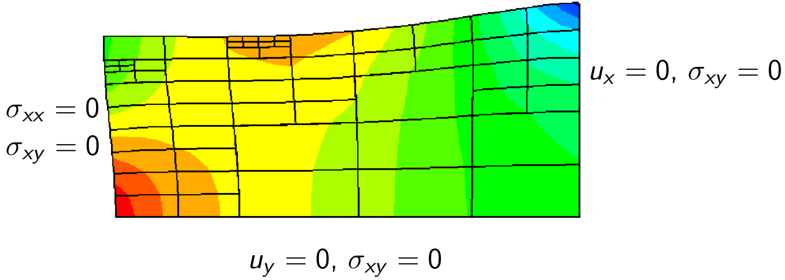

Summary of the method

- Choose type of element (pick an LBB stable one if incompressible problem).
- Choose a timestepper (must compute second derivatives).
- Generate mesh in undeformed solid domain.
- Specify boundary and initial conditions
  - For Dirichlet conditions, replace the discrete weak form.
  - For traction conditions, assemble the surface integral.
- Loop over elements and assemble the global residuals and Jacobian matrix for each time step
- Solve the (non)linear residual equations using Newton's method.
- Repeat for as many timesteps as desired.