Incompressible Fluid Flow

Andrew Hazel

Introduction

- The state of motion of a fluid is characterised by
  - velocity (vector) field $u_i\,(\mathbf{x}, t)$,
  - internal stress (tensor) field $\sigma_{ij}\,(\mathbf{x}, t)$,
  - density (scalar) field $\rho\,(\mathbf{x}, t)$.

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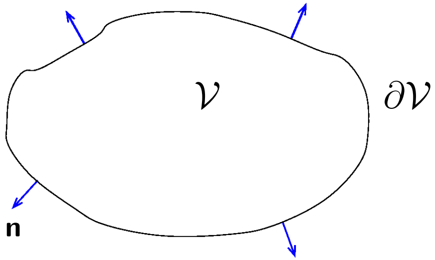
  - internal stress (tensor) field $\sigma_{ij}(\mathbf{x}, t)$,
  - density (scalar) field $\rho(\mathbf{x}, t)$.
- Consider conservation of physical quantities over a fixed control volume, $\mathcal{V}$, with outer unit normal, $\mathbf{n}$:



Conservation of mass

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathcal{V}} \rho \, \mathrm{d}V \quad + \quad \iint_{\partial\mathcal{V}} \rho u_j n_j \, \mathrm{d}S \quad = \quad \iiint_{\mathcal{V}} q \, \mathrm{d}V,$$

rate of change of mass $+$ outflux of mass $=$ total source of mass.

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- Equation must be satisfied for any control volume and so taking the infinitessimal limit gives

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \mathbf{u}) = q. \tag{1}$$

Conservation of momentum

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathcal{V}} \rho u_i \,\mathrm{d}V + \iint_{\partial\mathcal{V}} \rho u_i u_j n_j \,\mathrm{d}S = \iint_{\partial\mathcal{V}} \sigma_{ij} n_j \,\mathrm{d}S + \iiint_{\mathcal{V}} F_i \,\mathrm{d}V.$$

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- *Fixed* control volume and use of divergence theorem gives

$$\iiint_{\mathcal{V}} \frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} - \frac{\partial \sigma_{ij}}{\partial x_j} - F_i \,\mathrm{d}V = 0.$$

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- Taking the infinitessimal limit again leads to

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = \nabla \cdot \underline{\underline{\sigma}} + \mathbf{F}. \qquad (2)$$

Incompressible, Newtonian Fluid

- If the fluid is incompressible, density is constant and the equations (1) and (2) become

$$\rho\left(\frac{\partial u_i}{\partial t}+\frac{\partial(u_i u_j)}{\partial x_j}\right)=\frac{\partial \sigma_{ij}}{\partial x_j}+F_i \quad \text{and} \quad \frac{\partial u_j}{\partial x_j}=q.$$

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- For a Newtonian fluid the stress tensor has the form

$$\sigma_{ij}=-p\,\delta_{ij}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right),$$

where $p(\mathbf{x},t)$ is the fluid pressure.

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- For a Newtonian fluid the stress tensor has the form

$$\sigma_{ij} = -p\,\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

  where $p(\mathbf{x}, t)$ is the fluid pressure.

- Hence, the momentum equation becomes

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] + F_i.$$

Non-dimensionalisation

- We choose typical, length, velocity, pressure and time scales so that

$$x_i = L\,\overline{x}_i, \quad u_i = U\,\overline{u}_i, \quad p = P\,\overline{p} \quad \text{and} \quad t = T\,\overline{t},$$

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- The momentum equation becomes

$$\frac{\rho U}{T}\frac{\partial \overline{u}_i}{\partial \overline{t}} + \frac{\rho U^2}{L}\overline{u}_j\frac{\partial \overline{u}_i}{\partial \overline{x}_j} = -\frac{P}{L}\frac{\partial \overline{p}}{\partial \overline{x}_i} + \frac{\mu U}{L^2}\frac{\partial}{\partial \overline{x}_j}\left(\frac{\partial \overline{u}_i}{\partial \overline{x}_j} + \frac{\partial \overline{u}_j}{\partial \overline{x}_i}\right) + F_i.$$

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- If we choose to the pressure scale to be the viscous scale $P = \mu U/L$

$$\frac{\rho UL}{\mu}\left[\frac{L}{UT}\frac{\partial \overline{u}_i}{\partial \overline{t}} + \overline{u}_j\frac{\partial \overline{u}_i}{\partial \overline{x}_j}\right] = -\frac{\partial \overline{p}}{\partial \overline{x}_i} + \frac{\partial}{\partial \overline{x}_j}\left(\frac{\partial \overline{u}_i}{\partial \overline{x}_j} + \frac{\partial \overline{u}_j}{\partial \overline{x}_i}\right) + \frac{L^2}{\mu U}F_i.$$

Non-dimensionalisation

- In the absence of any body forces $F_i = 0$ and the momentum equation becomes (drop the overbars)

$$Re\left[St\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right] = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

  where

$$Re = \frac{\rho U L}{\mu} \quad \text{and} \quad St = \frac{L}{UT},$$

  the Reynolds and Strouhal numbers respectively.

- The conservation of mass equation remains essentially the same; and in the absence of any sources or sinks of mass

$$\frac{\partial u_j}{\partial x_j} = 0.$$

Finite Element Method

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  - What is the weak form of the Navier–Stokes equations?

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- When considering a finite element approach, the following questions naturally arise:
  - What is the weak form of the Navier–Stokes equations?
  - What basis functions should we choose for the velocity and pressure?
  - How do we impose boundary conditions?

Weak form of the Navier–Stokes equations

- Take the dot product of the momentum equation with a suitable vector-valued test function, ψ_i, and integrate

$$\iint \left\{ Re \left[St \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right] + \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right\} \psi_i \, \mathrm{d}V = 0.$$

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- Weaken the differentiability requirements by integrating the stress-tensor terms by parts

$$\iint Re\left[St\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right]\psi_i - p\frac{\partial \psi_i}{\partial x_i} + \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial \psi_i}{\partial x_j}\, \mathrm{d}V$$

$$= \iint \left[-pn_i + \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j \right]\psi_i \, \mathrm{d}S.$$

Surface traction

Weak form of the Navier–Stokes equation

- Weak form of the momentum equation

$$\iint Re\left[St\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right]\psi_i-p\frac{\partial \psi_i}{\partial x_i}+\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{\partial \psi_i}{\partial x_j}\,\mathrm{d}V$$
$$=\iint\left[-pn_i+\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)n_j\right]\psi_i\,\mathrm{d}S. \qquad (3)$$

- Multiply the continuity equation by a scalar test function, ϕ, and integrate

$$\iint \frac{\partial u_j}{\partial x_j}\,\phi\,\mathrm{d}V=0. \qquad (4)$$

- How do we choose the test functions?

Variational principle for the Stokes equations

- The Stokes equations are the Navier–Stokes equations in the limit $Re \to 0$

$$0 = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{and} \quad \frac{\partial u_j}{\partial x_j} = 0.$$

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- The solution of the Stokes equations satisfies a minimum dissipation theorem:
- Minimise

$$I(u_i) = \frac{1}{4}\iiint \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \mathrm{d}V,$$

subject to the constraint $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$.

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subject to the constraint $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$.

- Introducing a Lagrange multiplier λ leads to the functional

$$J(u_i, \lambda) = \iiint \frac{1}{4}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \lambda \frac{\partial u_j}{\partial x_j} \mathrm{d}V.$$

Variational principle for the Stokes equations

- Minimum is attained when $\delta J = 0$.
- Take variations with respect to u_i

$$\iiint \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \delta \frac{\partial u_i}{\partial x_j} - \lambda \delta \frac{\partial u_j}{\partial x_j} \, \mathrm{d}V = 0.$$

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$$\Rightarrow \iiint -\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\delta u_i + \frac{\partial \lambda}{\partial x_j}\delta u_j\,\mathrm{d}V = 0.$$

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- Must be true for all variations and all control volumes, so

$$-\frac{\partial \lambda}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = 0.$$

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as enforced by the Lagrange multiplier method.

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- Velocity and Lagrange multiplier fields satisfy the Stokes equations

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- Moreover, we identify the pressure as the Lagrange multiplier that enforces the divergence-free constraint.

Relating the weak form and variational principle

- Compare the weak form and first variation

$$\iiint \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \delta\frac{\partial u_i}{\partial x_j} - \lambda\delta\frac{\partial u_j}{\partial x_j}\,\mathrm{d}V = 0 \text{ and } \iiint \frac{\partial u_j}{\partial x_j}\,\delta\lambda\,\mathrm{d}V = 0.$$

$$\iiint \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \frac{\partial \psi_i}{\partial x_j} - p\,\frac{\partial \psi_j}{\partial x_j}\,\mathrm{d}V = 0 \text{ and } \iiint \frac{\partial u_j}{\partial x_j}\,\phi\,\mathrm{d}V = 0.$$

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- Test functions ψ_i are associated with velocity.
- Test functions ϕ are associated with pressure.

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- Test functions ψ_i are associated with velocity.
- Test functions ϕ are associated with pressure.
- For Galerkin method: Choose ψ_i to be the fluid velocity basis functions and ϕ to be the fluid pressure basis functions.

Finite Elements: The Basic Method

- Approximate velocity and pressure by sums of discrete basis functions

$$\mathbf{u}(\mathbf{x},t) = \sum_{l=0}^{3N_u} u_l(t)\psi_l(\mathbf{x}), \quad p(\mathbf{x},t) = \sum_{l=0}^{N_p} p_l(t)\phi_l(\mathbf{x}).$$

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- It is very common to choose the vector basis functions to be scalar basis functions in a specific coordinate directions

$$\boldsymbol{\psi}_l = \begin{cases} (\psi_l, 0, 0) & l \leq N_u \\ (0, \psi_{l-N_u}, 0) & N_u < l \leq 2N_u \\ (0, 0, \psi_{l-2N_u}) & l > 2N_u \end{cases}$$

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- Choose the test functions in (3) to be the velocity basis and in (4) to be the pressure basis.

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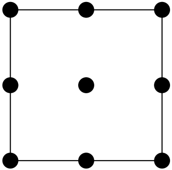
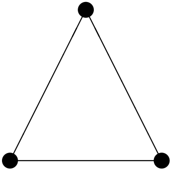
- Choose the test functions in (3) to be the velocity basis and in (4) to be the pressure basis.
- Assemble the set of discrete nonlinear residual equations for the discrete unknowns $\{u_I, p_I\}$.

Choosing the basis functions

- **Idea:** Pick low order polynomials with compact support.
- Divide space into elements
  - quadrilaterals or triangles (2D)
  - hexahedrons or tetrahedrons (3D)
- Define elements by a number of nodes $\mathbf{x}_i$ and insist that $\psi_I(\mathbf{x}_j) = \delta_{Ij}$.



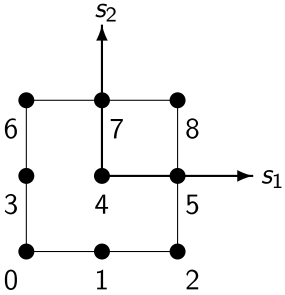
Quadratic Quadrilateral



Linear Triangle

Local coordinates

- Introduce local coordinates, $\mathbf{s}$, within each element.
- Basis functions are easy(ish) in local coordinates.
- Must introduce local node numbering scheme (document it!)
- Must decide on range of local coordiantes, *e.g.* $s_i \in [-1, 1]$



$$\psi_0(s_1, s_2) = \frac{1}{4} s_1 s_2 (1 - s_1)(1 - s_2)$$

$$\psi_1(s_1, s_2) = \frac{1}{2} s_2 (1 - s_1^2)(1 - s_2)$$

$$\psi_2(s_1, s_2) = \frac{1}{4} s_2 s_1 (1 + s_1)(1 - s_2)$$

$$\vdots$$

- Look up common basis functions in books and papers.

Weak form of Stokes equations

- In each element we must assemble

$$\iiint \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial \psi_l}{\partial x_j} - p \, \frac{\partial \psi_l}{\partial x_j} \, \mathrm{d}V = 0 \text{ and } \iiint \frac{\partial u_j}{\partial x_j} \, \phi_l \, \mathrm{d}V = 0.$$

- If we use the same scalar basis functions for each velocity component then within the element

$$u_i(\mathbf{x}) = \sum_{j=0}^{N_u-1} \hat{u}_{ij} \psi_j(\mathbf{s}) \quad \text{and} \quad p(\mathbf{x}) = \sum_{j=0}^{N_p-1} \hat{p}_j \phi_j(\mathbf{s}),$$

where $\hat{u}_{ij}$ is the i-th velocity component at the j-th node and $\hat{p}_j$ is the j-th pressure unknown.

Computing derivatives of the unkowns

- Introduction of local coordinates means that we must use the chain rule to take spatial derivatives

$$\frac{\partial u_i}{\partial x_j} = \sum_l \hat{u}_{il} \frac{\partial \psi_l\,(s_k)}{\partial x_j} = \sum_l \hat{u}_{il} \frac{\partial \psi_l}{\partial s_m} \frac{\partial s_m}{\partial x_j}.$$

- Computation of the derivative $\partial s_m / \partial x_j$ requires an inversion of the mapping from local to global coordinates $x_j(s_m)$.
- What is the mapping?
- **Idea:** Use the same basis functions as the velocity components to describe the mapping from local to global coordinates — an isoparametric mapping.

$$x_i = \sum_j \hat{x}_{ij} \psi_j(s_k),$$

where $\hat{x}_{ij}$ is the i-th coordinate of the j-th node.

Integrating over the element

- Can compute the integrand at any point within the element.
- Integrate using numerical quadrature — a Gauss rule.

$$\int_{x_1}^{x_2} f(x)\,\mathrm{d}x = \int_{-1}^{1} f(x(s))\frac{\mathrm{d}x}{\mathrm{d}s}\,\mathrm{d}s \approx \sum_{ipt=1}^{N_g} w_{ipt} f(s_{ipt}) J(s_{ipt}).$$

- $J(s) = x'(s)$ is the Jacobian of the mapping from local coordinates to global coordinates.
- The order of the Gauss rule should be chosen so that the integral is exact for the order of polynomial chosen for the basis functions.
- Look up common Gauss rules in books and papers.

Elemental residuals

- Define the residuals for each element

$$R^{(u)}_{l,i} = \sum_{ipt} \left[\left(\frac{\partial u_i}{\partial x_j}(s_{ipt}) + \frac{\partial u_j}{\partial x_i}(s_{ipt}) \right) \frac{\partial \psi_l}{\partial x_j}(s_{ipt}) \right.$$

$$\left. - p(s_{ipt}) \, \frac{\partial \psi_l}{\partial x_i}(s_{ipt}) \right] J(s_{ipt}) w_{ipt},$$

$$R^{(p)}_{l} = \sum_{ipt} \frac{\partial u_j}{\partial x_j}(s_{ipt}) \phi_l(s_{ipt}) J(s_{ipt}) w_{ipt},$$

- To assemble loop over all Gauss points, compute the required shape functions and derivatives, Jacobian and add the contribution to each residual.

Global Assembly

- We can now assemble the *local* residuals in each element
- How do we form a *global* system of equations?
- Loop over all elements and add contributions to the "appropriate" *global* residuals.
- Need a translation from local node numbers to global node numbers.
- Setting up this translation scheme is the job of the mesh generator.
- Information must be stored somewhere in your program.

The global residual problem

- For the 2D Stokes equations, once we have an global numbering for the unknowns, the linear system can be written in the form

$$\mathcal{R}(\mathcal{U}) = \begin{pmatrix} K_{11} & K_{12} & C_1 \\ K_{21} & K_{22} & C_2 \\ C_1^T & C_2^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{p} \end{pmatrix} = \mathbf{0}.$$

$$K_{ij} = \int \frac{\partial \psi}{\partial x_i} \otimes \frac{\partial \psi}{\partial x_j}, \, \mathrm{d}V \quad C_i = \int \phi \otimes \frac{\partial \psi}{\partial x_i} \, \mathrm{d}V,$$

- $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{p}$ are vectors of global velocity and pressure unknowns.
- ψ and ϕ are vectors of (global) basis functions arranged in order of the unknowns.
- A linear algebra problem!

Newton's method for Navier–Stokes

- For Navier–Stokes, get additional term in the momentum equations — nonlinear residuals

$$\mathcal{R}(\mathcal{U}) = \begin{pmatrix} K_{11} + Re\,N(\mathbf{u}) & K_{12} & C_1 \\ K_{21} & K_{22} + Re\,N(\mathbf{u}) & C_2 \\ C_1^T & C_2^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{p} \end{pmatrix} = \mathbf{0},$$

$$\text{where} \quad N(\mathbf{u}) = \iiint u_j \frac{\partial \boldsymbol{\psi}}{\partial x_j} \otimes \boldsymbol{\psi}\, \mathrm{d}V.$$

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- Use Newton's method to find solution $\mathcal{R}_j(\mathcal{U}_i) = 0$

Newton's method for Navier–Stokes

- Use Newton's method to find solution $\mathcal{R}_j(\mathcal{U}_i) = 0$
- Assume that we have an initial solution $\mathcal{U}_i^{(0)}$ and add a correction $\delta U_i^{(0)}$ to get the exact solution

$$\mathcal{R}_j(\mathcal{U}_i^{(0)} + \delta U_i^{(0)}) = 0.$$

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- A Taylor-series expansion gives

$$\mathcal{R}_j(\mathcal{U}_i^{(0)}) + \frac{\partial \mathcal{R}_j}{\partial \mathcal{U}_k}(\mathcal{U}_i^{(0)})\, \delta U_k^{(0)} \approx 0.$$

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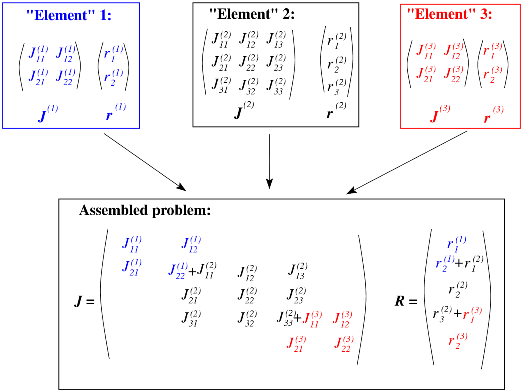
- To find correction solve the linear system

$$\mathcal{J}\,\boldsymbol{\delta}\mathbf{U} = -\boldsymbol{\mathcal{R}}, \quad \text{where} \quad \mathcal{J}_{ij} = \frac{\partial \mathcal{R}_i}{\partial \mathcal{U}_j}.$$

- Converges quadratically for good initial guess (can always start from $Re = 0$)

Newton's method for Navier–Stokes

- Can assemble *local* (elemental) Jacobian matrix and residuals
- Loop over all elements and add contributions to *global* Jacobian matrix and residuals vector.



- Can always approximate $\mathcal{J}$ by finite differences.

Choice of velocity and pressure basis functions

$$\iiint Re\, u_j \frac{\partial u_i}{\partial x_j}\psi_i + \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial \psi_i}{\partial x_j} - p\,\frac{\partial \psi_j}{\partial x_j}\,\mathrm{d}V = 0$$

$$\text{and}\quad \iiint \frac{\partial u_j}{\partial x_j}\,\phi\,\mathrm{d}V = 0.$$

- Are we restricted in our choice of basis functions?
- **Yes!** Only certain combinations of basis functions lead to "good" (stable) elements.

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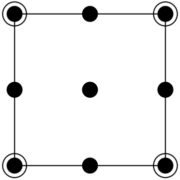
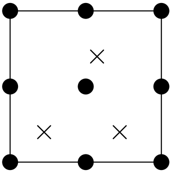
- Equal-order interpolation ($\psi = \phi$) is always a bad choice.
- Weak form indicates that ψ higher differentiability requirements than ϕ.
- Suggests that ψ should be higher order than ϕ.
- General theory (*inf-sup*/LBB stability) exists, but is difficult.

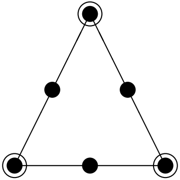
Some LBB-stable elements



$Q_2 Q_1$ (Taylor–Hood)
Quadratic velocity
Linear pressure



$Q_2 P_{-1}$
Quadratic velocity
Linear pressure



$P_2 P_1$ (Taylor–Hood)
Quadratic velocity
Linear pressure

- Pressure need not be continuous between elements.
- Discontinuous pressure leads to load mass conservation because for piecewise constant test function ϕ

$$\iiint_E \phi \boldsymbol{\nabla} \cdot \mathbf{u} \, \mathrm{d}V = 0 \quad \Rightarrow \quad \iiint_E \boldsymbol{\nabla} \cdot \mathbf{u} \, \mathrm{d}V = 0.$$

Boundary conditions

- Recall the weak form of the Navier–Stokes equations

$$\iiint Re\left[St\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right]\psi_i - p\frac{\partial \psi_i}{\partial x_i}+\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{\partial \psi_i}{\partial x_j}\,\mathrm{d}V$$

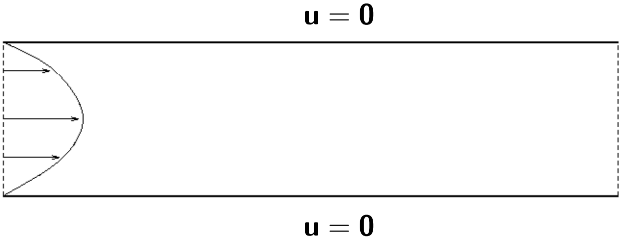
$$=\iint\left[-pn_i+\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)n_j\right]\psi_i\,\mathrm{d}S,$$

$$\iint\frac{\partial u_j}{\partial x_j}\,\phi\,\mathrm{d}V=0.$$

- What are the allowed boundary conditions?
  - Specified velocity on boundary.
  - Specified traction on boundary.
- If the surface integral is not included then (implicitly) the traction is set to zero.

Example problem: Poiseuille Flow

- Steady flow in channel has parabolic profile, $\mathbf{u}_p = (u_p(y), 0)$.
- What are appropriate boundary conditions?



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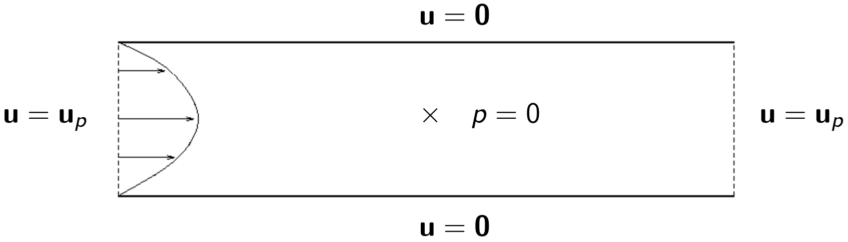

- Prescribe inlet and outlet velocity
  - Dangerous ... must have discretely divergence free boundary conditions.
  - Must choose a reference pressure value otherwise problem is underdetermined.

Example problem: Poiseuille Flow

- Steady flow in channel has parabolic profile, $\mathbf{u}_p = (u_p(y), 0)$.
- What are appropriate boundary conditions?

$$\mathbf{u} = \mathbf{0}$$

$\mathbf{u} = \mathbf{u}_p$

$\sigma_{ij} n_j = 0$

$$\mathbf{u} = \mathbf{0}$$

- Prescribe inlet and leave outlet traction free.
  - At outlet

$$\int -p + 2\frac{\partial u}{\partial x}\, \mathrm{d}y = 0, \quad \int \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\, \mathrm{d}y = 0,$$

Example problem: Poiseuille Flow

- Steady flow in channel has parabolic profile, $\mathbf{u}_p = (u_p(y), 0)$.
- What are appropriate boundary conditions?

- Prescribe inlet and leave outlet traction free.
  - At outlet
$$\int p \, \mathrm{d}y = 0, \quad \int \frac{\partial u}{\partial y} \, \mathrm{d}y = \bar{u_p} = 0,$$
  - Implicitly sets average pressure to zero.
  - Implicitly sets average velocity to zero (contradiction).

Example problem: Poiseuille Flow

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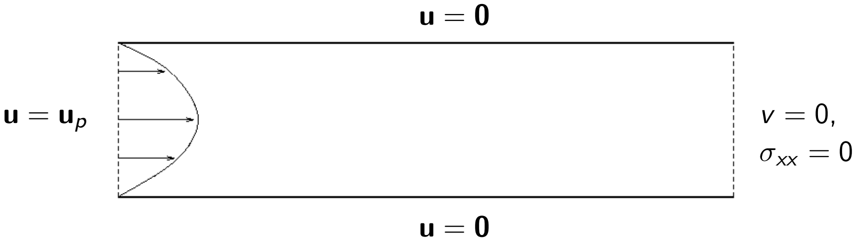

- Prescribe inlet, set $v = 0$ and $\sigma_{xx} = 0$ at outlet
  - At outlet

$$v = 0, \quad \int -p + \frac{\partial u}{\partial x}\, \mathrm{d}y = \int_{y_1}^{y_2} p\, \mathrm{d}y = [v]_{y_1}^{y_2} = 0.$$

  - Here, $p = 0$ pointwise at the outlet.

Example problem: Poiseuille Flow

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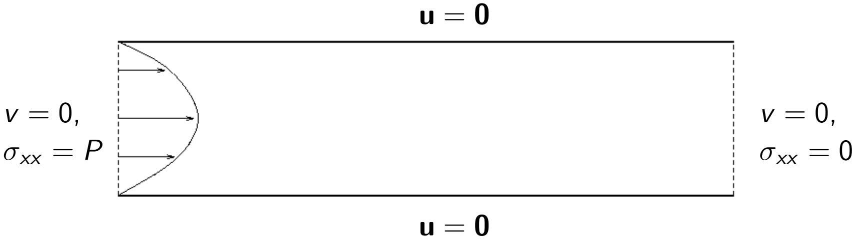
- What are appropriate boundary conditions?



- Prescribe inlet and outlet pressures
  - Equivalent to imposing pressure drop.
  - Can trade imposed pressure for desired flow rate Q.

Example problem: Poiseuille Flow

- Steady flow in channel has parabolic profile, $\mathbf{u}_p = (u_p(y), 0)$.
- What are appropriate boundary conditions?

$$\mathbf{u} = \mathbf{0}$$

$$\mathbf{u} = \mathbf{0}$$

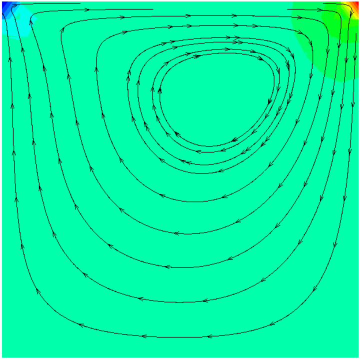
- Can prescribe flow rate, pressure drop or velocity profile.
- *c.f.* Physical experiment.

Example Problem: Driven Cavity

- Enclosed flow, so pressure must be specified somewhere.

$\mathbf{u} = (1, 0)$

$\mathbf{u} = \mathbf{0}$



$\mathbf{u} = \mathbf{0}$

$Re = 100$

$\mathbf{u} = \mathbf{0}$

- Pressure singularities in corners, handled “better” by discontinuous pressure interpolation.

Summary of the method for steady flows

- Choose type of element (pick an LBB stable one initially).
- Generate mesh in flow domain.
- Specify boundary conditions
  - For Dirichlet conditions, replace the discrete weak form.
  - For traction conditions, assemble the surface integral.
- Loop over elements and assemble the global residuals and Jacobian matrix.
- Solve the (non)linear residual equations using Newton's method.

Unsteady flows

- Need to approximate the time derivative, $\partial \mathbf{u} / \partial t$
- No time-derivative of pressure!
- Explicit, or semi-explicit, methods require a projection step (or equivalent) to ensure that the velocity field is divergence free.
- Implicit methods do not, but require the solution of a fully-coupled problem ...
- ... no harder than solving for a steady problem.
(In fact, matrices are often better conditioned because of addition of mass matrix to diagonal).

Unsteady flows

- Use a second order backward differentiation formula (BDF2) to approximate the time derivative

$$\left[\frac{\partial \mathbf{u}}{\partial t}\right]^{n+1} \approx \frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^{n} + \mathbf{u}^{n-1}}{2\Delta t}$$

- Terms involving history values $\mathbf{u}^n$ and $\mathbf{u}^{n-1}$ move onto RHS.
- Additional diagonal matrices are mass matrices

$$ReSt\frac{3}{2\Delta t}\iiint \psi \otimes \psi \, \mathrm{d}V.$$

- Assemble and solve just as for steady problem.
- Need additional storage for history values.

Summary of the method for unsteady flows

- Choose type of element (pick an LBB stable one initially).
- Choose a timestepper.
- Generate mesh in flow domain.
- Specify boundary and initial conditions
  - For Dirichlet conditions, replace the discrete weak form.
  - For traction conditions, assemble the surface integral.
- Loop over elements and assemble the global residuals and Jacobian matrix for each time step
- Solve the (non)linear residual equations using Newton's method.
- Repeat for as many timesteps as desired.