MATH45061: SOLUTION SHEET[1] IV

1.) The rate of change of mass of the deformed region must be equal to the total mass production rate of the region

$$\frac{D}{Dt}\int_{\Omega_t} \rho \,\mathrm{d}\mathcal{V}_t = \int_{\Omega_t} \rho\gamma \,\mathrm{d}\mathcal{V}_t. \tag{1}$$

Using the Reynolds transport theorem we obtain

$$\int_{\Omega_t} \left[\frac{D\rho}{Dt} + \rho \boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V} - \rho\gamma\right] \mathrm{d}\mathcal{V}_t = 0;$$

and by the usual argument that because the relationship must be true for any volume it must be true pointwise, we have the desired equation

$$\frac{D\rho}{Dt} + \rho \boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V} = \rho\gamma.$$

The Lagrangian form is actually more complicated. We have that the total mass is equal to the initial mass plus the additional mass from growth over the time interval, so

$$\int_{\Omega_t} \rho \,\mathrm{d}\mathcal{V}_t = \int_{\Omega_0} \rho_0 \,\mathrm{d}\mathcal{V}_0 + \int_0^t \int_{\Omega_s} \rho\gamma \,\mathrm{d}\mathcal{V}_s \,\mathrm{d}s,$$

and converting all integrals to the undeformed configuration gives

$$\int_{\Omega_0} \rho J(t) \,\mathrm{d}\mathcal{V}_0 = \int_{\Omega_0} \rho_0 \,\mathrm{d}\mathcal{V}_0 + \int_0^t \int_{\Omega_0} \rho(s)\,\gamma(s) J(s) \,\mathrm{d}\mathcal{V}_0 \,\mathrm{d}s.$$

The initial domain is fixed so we can commute the integrals and because the initial volume is arbitrary we can write our final equation

$$\rho J(t) = \rho_0 + \int_0^t \rho(s)\,\gamma(s) J(s)\,\mathrm{d}s,$$

or

$$\rho J - \int_0^t \rho(s)\,\gamma(s) J(s)\,\mathrm{d}s = \rho_0.$$

This is a more painful equation to deal with because we have to keep track of the entire past history of the motion and we have to solve an integral equation to find ρ. If we had specified the growth rate per unit volume $G = \rho\gamma$ then we can calculate ρ explicitly if we know the deformation history

$$\rho = \frac{1}{J(s)}\left[\rho_0 + \int_0^t G(s) J(s)\,\mathrm{d}s\right].$$

[1]Any feedback to: *Andrew.Hazel@manchester.ac.uk*

2.) The easiest solution here is the "or otherwise". From the definition of the first Piola–Kirchhoff stress tensor

$$\mathsf{p} = J\sigma^T \mathsf{F}^{-T}, \tag{2}$$

and so

$$\mathsf{p}\mathsf{F}^T = J\sigma^T. \tag{3}$$

Taking the transpose of equation (2) we obtain

$$\mathsf{p}^T = (J\sigma^T \mathsf{F}^{-T})^T = J\mathsf{F}^{-1}\sigma,$$

and so

$$\mathsf{F}\mathsf{p}^T = J\sigma = J\sigma^T, \tag{4}$$

because the Cauchy stress tensor is symmetric. Hence, from equations (2) and (4)

$$\mathsf{p}\mathsf{F}^T = \mathsf{F}\mathsf{p}^T,$$

which demonstrates that the first Piola–Kirchhoff stress tensor is not symmetric and

$$\mathsf{p} = \mathsf{F}\mathsf{p}^T\mathsf{F}^{-T}.$$

3.) The correct form of the stress power to use here is that involving the second Piola–Kirchhoff stress

$$\dot{W} = \mathsf{s} : \dot{\mathsf{e}} = s_{IJ}\dot{e}_{IJ}.$$

For the rigid motion $X_K = A_{KJ}x_J + C_K$ and

$$e_{IJ} = \frac{1}{2}\left(\frac{\partial X_K}{\partial x_I}\frac{\partial X_K}{\partial x_J} - \delta_{IJ}\right) = \frac{1}{2}\left(A_{KJ}A_{KI} - \delta_{IJ}\right) = 0,$$

because A is orthogonal, so $A_{KJ}A_{KI} = \delta_{IJ}$. Hence $\dot{\mathsf{e}} = 0$ and the stress power per unit undeformed volume is zero, which, of course, implies that the stress power per unit deformed volume is zero.

4.) From the lecture notes we have that

$$\dot{W} = \int_{\Omega_0} s^{ij}\dot{\gamma}_{ij}\,\mathrm{d}\mathcal{V}_0 = \int_{\Omega_0} s_{IJ}\dot{e}_{IJ}\,\mathrm{d}\mathcal{V}_0 = \int_{\Omega_0} F^{-1}_{IK}p_{KJ}\frac{D}{Dt}\left[\frac{1}{2}\left(F_{LI}F_{LJ} - \delta_{IJ}\right)\right]\,\mathrm{d}\mathcal{V}_0,$$

using the definition of e_{IJ}. Thus,

$$\begin{aligned}\dot{W} &= \int_{\Omega_0} F^{-1}_{IK}p_{KJ}\frac{1}{2}\left(\dot{F}_{LI}F_{LJ} + F_{LI}\dot{F}_{LJ}\right)\,\mathrm{d}\mathcal{V}_0 \\ &= \frac{1}{2}\int_{\Omega_0} F_{LI}F^{-1}_{IK}p_{KJ}\dot{F}_{LJ} + F^{-1}_{IK}p_{KJ}\dot{F}_{LI}F_{LJ}\,\mathrm{d}\mathcal{V}_0.\end{aligned}$$

We know that $s_{IJ} = s_{JI}$, so $F^{-1}_{IK}p_{KJ} = F^{-1}_{JK}p_{KI}$ (which is a variation of the result in question 2.) and therefore

$$\begin{aligned}\dot{W} &= \frac{1}{2}\int_{\Omega_0} F_{LI}F^{-1}_{IK}p_{KJ}\dot{F}_{LJ} + F_{LJ}F^{-1}_{JK}p_{KI}\dot{F}_{LI}\,\mathrm{d}\mathcal{V}_0 = \int_{\Omega_0}\delta_{LK}p_{KJ}\dot{F}_{LJ}\,\mathrm{d}\mathcal{V}_0 \\ &= \int_{\Omega_0} p_{LJ}\dot{F}_{LJ}\,\mathrm{d}\mathcal{V}_0 = \int_{\Omega_0}\mathsf{p} : \dot{\mathsf{F}}\,\mathrm{d}\mathcal{V}_0,\end{aligned}$$

which means that the stress power per undeformed unit volume is $\mathsf{p} : \dot{\mathsf{F}}$.

5.) The general equations of conservation of mass, momentum and energy are given by

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}_{\!\boldsymbol{R}} \boldsymbol{\cdot} (\rho \boldsymbol{V}) = 0, \tag{5}$$

$$\rho \frac{D\boldsymbol{V}}{Dt} = \boldsymbol{\nabla}_{\!\boldsymbol{R}} \boldsymbol{\cdot} \mathsf{T} + \rho \boldsymbol{F}, \tag{6}$$

and

$$\rho \frac{D\Phi}{Dt} = \mathsf{T} : \mathsf{D} + \rho B - \boldsymbol{\nabla}_{\!\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{Q}. \tag{7}$$

The constitutive assumption will not change the conservation of mass equation, which is purely kinematic, so we don't have to consider it further. Under the assumption that $\mathsf{T} = -P(\boldsymbol{R}, t)\, \mathsf{I}$,

$$\boldsymbol{\nabla}_{\!\boldsymbol{R}} \boldsymbol{\cdot} \mathsf{T} = -\boldsymbol{\nabla}_{\!\boldsymbol{R}} \boldsymbol{\cdot} (P\, \mathsf{I}).$$

In addition,

$$\mathsf{T} : \mathsf{D} = -P\, \mathsf{I} : \mathsf{D} = -P\, \delta^{\bar{i}}_{\bar{j}}\, D^{\bar{j}}_{\bar{i}} = -P\, D^{\bar{i}}_{\bar{i}} = -P\, V^{\bar{i}}||_{\bar{i}} = -P\, \boldsymbol{\nabla}_{\!\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V}.$$

Expanding the material derivative the momentum equation (6) becomes

$$\rho \frac{\partial \boldsymbol{V}}{\partial t} + \rho \boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{\nabla}_{\!\boldsymbol{R}} \boldsymbol{V} + \boldsymbol{\nabla}_{\!\boldsymbol{R}} \boldsymbol{\cdot} (\boldsymbol{P} \mathsf{I}) = \rho \boldsymbol{F},$$

$$\Rightarrow \frac{\partial \rho \boldsymbol{V}}{\partial t} - \boldsymbol{V} \frac{\partial \rho}{\partial t} + \rho \boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{\nabla}_{\!\boldsymbol{R}} \boldsymbol{V} + \boldsymbol{\nabla}_{\!\boldsymbol{R}} \boldsymbol{\cdot} (\boldsymbol{P} \mathsf{I}) = \rho \boldsymbol{F},$$

and using the equation of conservation of mass

$$\frac{\partial \rho \boldsymbol{V}}{\partial t} + \boldsymbol{V} \boldsymbol{\nabla}_{\!\boldsymbol{R}} \boldsymbol{\cdot} (\rho \boldsymbol{V}) + \rho \boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{\nabla}_{\!\boldsymbol{R}} \boldsymbol{V} + \boldsymbol{\nabla}_{\!\boldsymbol{R}} \boldsymbol{\cdot} (\boldsymbol{P} \mathsf{I}) = \rho \boldsymbol{F}.$$

In index notation the second and third terms are

$$V_I (\rho V_J)_{,J} + \rho V_J V_{I,J} = (\rho V_I V_J)_{,J},$$

using the product rule, so returning to dyadic notation we have

$$\frac{\partial \rho \boldsymbol{V}}{\partial t} + \boldsymbol{\nabla}_{\!\boldsymbol{R}} \boldsymbol{\cdot} \left(\rho \boldsymbol{V} \otimes \boldsymbol{V} + P\, \mathsf{I}\right) = \rho \boldsymbol{F},$$

as required.

Finally, for the energy equation (7) we have

$$\rho \frac{\partial \Phi}{\partial t} + \rho \boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{\nabla}_{\boldsymbol{R}} \Phi + P\, \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V} = \rho B - \boldsymbol{\nabla}_{\!\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{Q}.$$

Expanding the first derivative and using conservation of mass, as above, we have

$$\frac{\partial \rho \Phi}{\partial t} + \Phi \boldsymbol{\nabla}_{\!\boldsymbol{R}} \boldsymbol{\cdot} (\rho \boldsymbol{V}) + \rho \boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{\nabla}_{\boldsymbol{R}} \Phi + P\, \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V} = \rho B - \boldsymbol{\nabla}_{\!\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{Q}.$$

$$\Rightarrow \frac{\partial \rho\Phi}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} (\rho\Phi \boldsymbol{V}) + P\, \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V} = \rho B - \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{Q}, \tag{8}$$

by using the product rule. The pressure term in the energy equation (8) is

$$P\, \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V} = \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} (P \boldsymbol{V}) - \boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{\nabla}_{\boldsymbol{R}} P; \tag{9}$$

and we can find an expression for $\boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{\nabla}_{\boldsymbol{R}} P$ from the momentum equation (6), which can be written as

$$\rho \frac{D\boldsymbol{V}}{Dt} + \boldsymbol{\nabla}_{\boldsymbol{R}} P = \rho \boldsymbol{F},$$

because

$$[\boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} (P\mathsf{I})]_{\bar{i}} = -(P\delta^{\bar{j}}_{\bar{i}})||_{\bar{j}} = -P||_{\bar{j}}\delta^{\bar{j}}_{\bar{i}} = -P||_{\bar{i}} = -\frac{\partial P}{\partial \chi^{\bar{i}}} = -\left[\boldsymbol{\nabla}_{\boldsymbol{R}} P\right]_{\bar{i}}.$$

(I'm showing off slightly, this calculation is most easily done in Cartesians, as it was in lectures.) Taking the dot product of the momentum equation with $\boldsymbol{V}$ gives

$$\rho \boldsymbol{V} \boldsymbol{\cdot} \frac{D\boldsymbol{V}}{Dt} - \rho \boldsymbol{F} \boldsymbol{\cdot} \boldsymbol{V} = -\boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{\nabla}_{\boldsymbol{R}} P,$$

$$\Rightarrow \frac{D}{Dt}\left(\frac{1}{2}\rho \boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{V}\right) - \frac{1}{2}\boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{V} \frac{D\rho}{Dt} - \rho \boldsymbol{F} \boldsymbol{\cdot} \boldsymbol{V} = -\boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{\nabla}_{\boldsymbol{R}} P,$$

and expanding the first material derivative and using the conservation of mass for the second gives

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{1}{2}\rho \boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{V}\right) + \boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{\nabla}_{\boldsymbol{R}} \left(\frac{1}{2}\rho \boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{V}\right) + \frac{1}{2}\boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{V} \rho \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V} - \rho \boldsymbol{F} \boldsymbol{\cdot} \boldsymbol{V} = -\boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{\nabla}_{\boldsymbol{R}} P,$$

and using the product rule

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{1}{2}\rho |\boldsymbol{V}^2\right) + \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \left(\frac{1}{2}\rho |\boldsymbol{V}|^2\, \boldsymbol{V}\right) - \rho \boldsymbol{F} \boldsymbol{\cdot} \boldsymbol{V} = -\boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{\nabla}_{\boldsymbol{R}} P. \tag{10}$$

Thus using equation (10) and equation (9) in equation (8) we obtain

$$\frac{\partial \rho\Phi}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} (\rho\Phi \boldsymbol{V}) + \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} (P \boldsymbol{V})$$

$$+ \frac{\partial}{\partial t}\left(\frac{1}{2}\rho |\boldsymbol{V}|^2\right) + \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \left(\frac{1}{2}\rho |\boldsymbol{V}|^2\, \boldsymbol{V}\right) - \rho \boldsymbol{F} \boldsymbol{\cdot} \boldsymbol{V} = \rho B - \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{Q},$$

and so

$$\frac{\partial \rho \left(\Phi + \frac{1}{2}|\boldsymbol{V}|^2\right)}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \left(\left[\rho\Phi + \frac{1}{2}\rho |\boldsymbol{V}|^2 + P\right] \boldsymbol{V}\right) = \rho B - \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{Q} + \rho \boldsymbol{F} \boldsymbol{\cdot} \boldsymbol{V};$$

which is the desired expression

$$\frac{\partial \rho E}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \left([\rho E + P] \boldsymbol{V}\right) = \rho B - \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{Q} + \rho \boldsymbol{F} \boldsymbol{\cdot} \boldsymbol{V},$$

where $E = \Phi + \frac{1}{2}|\boldsymbol{V}|^2$. In other words the total energy of the material is the internal energy plus the kinetic energy.

6.) From the second law of thermodynamics

$$D - \frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!\boldsymbol{R}}\Theta \geq 0.$$

In the absence of dissipation we have that

$$-\frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!\boldsymbol{R}}\Theta \geq 0,$$

and because $\boldsymbol{\Theta} > 0$

$$\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!\boldsymbol{R}}\Theta \leq 0.$$

If $\boldsymbol{Q}$ is a constant vector then the term $\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!\boldsymbol{R}}$ is simply the derivative in the direction $\boldsymbol{Q}$. Thus, the gradient of temperature in the direction of $\boldsymbol{Q}$ must be negative.

If there is dissipation, but $\dot{\eta} = 0$, then (from the lecture notes)

$$D = \boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{Q} - \rho B.$$

The heat flux $\boldsymbol{Q}$ is constant so its gradient is zero and $D = -\rho B$. Thus the second law becomes

$$-\rho B - \frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!\boldsymbol{R}}\Theta \geq 0,$$
$$\Rightarrow \quad \boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!\boldsymbol{R}}\Theta \leq -\rho\Theta B$$

The density and temperature are always positive, so the temperature gradient in the direction of the heat flux can be made positive if B is negative. In other words, if there is a body sink of heat, the temperature gradient can be positive in the direction of the heat flux. Note that just because positive gradients are not thermodynamically forbidden doesn't mean that they will occur. However, the larger the magnitude of the heat sink, the larger the range of allowed positive gradients.

7.) **a.)** The constitutive relations state that the heat flux per unit undeformed area is proportional to the gradient of temperature (this is Fourier's law) and that the internal energy per unit mass is proportional to the temperature.

b.) The conservation of energy equation in Lagrangian form is

$$\rho_0\frac{\partial\phi}{\partial t} = s^{ij}\dot{\gamma}_{ij} + \rho_0 b - \boldsymbol{\nabla}_{\!\boldsymbol{r}}\boldsymbol{\cdot}\boldsymbol{q}.$$

The body is undergoing rigid body motions only, so there is no stress power (see question 3) because there is never any strain. Thus,

$$\rho_0\frac{\partial\phi}{\partial t} = +\rho_0 b - \boldsymbol{\nabla}_{\!\boldsymbol{r}}\boldsymbol{\cdot}\boldsymbol{q},$$
$$\Rightarrow \quad \rho_0\alpha\frac{\partial\theta}{\partial t} = \rho_0 b + \kappa\boldsymbol{\nabla}_{\!\boldsymbol{r}}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!\boldsymbol{r}}\theta.$$

Dividing by $\rho_0\alpha$ gives the required heat equation

$$\frac{\partial\theta}{\partial t} = \frac{\kappa}{\rho_0\alpha}\boldsymbol{\nabla}_{\!\boldsymbol{r}}^2\theta + \frac{b}{\alpha} \quad \Rightarrow \quad \frac{\partial\theta}{\partial t} = D\boldsymbol{\nabla}_{\!\boldsymbol{r}}^2\theta + S,$$

where $D = \kappa/(\rho_0\alpha)$, $S = b/\alpha$.

c.) The second law of thermodynamics in terms of the free energy is given by

$$-\rho_0\dot{\psi} - \rho_0\eta_0\dot{\theta} - \frac{1}{\theta}\boldsymbol{q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!r}\theta + \mathsf{S} : \dot{\mathsf{e}} \geq 0.$$

There is no stress power and $\boldsymbol{q} = -\kappa\boldsymbol{\nabla}_{\!r}\theta$. If ψ and η_0 are functions only of θ then $\dot{\psi} = \frac{\partial\psi}{\partial\theta}\dot{\theta}$ and so the second law becomes

$$\left[-\rho_0\frac{\partial\psi}{\partial\theta} - \rho_0\eta_0\right]\dot{\theta} + \frac{\kappa}{\theta}|\boldsymbol{\nabla}_{\!r}\theta|^2 \geq 0.$$

If this condition must be satisfied for general thermal processes then from considering uniform temperature fields that increase or decrease in time, we have the constraint that

$$-\frac{\partial\psi}{\partial\theta} = \eta_0.$$

Given that this constraint is satisfied then

$$\frac{\kappa}{\theta}|\boldsymbol{\nabla}_{\!r}\theta|^2 \geq 0,$$

and because $\theta > 0$ and $|\boldsymbol{\nabla}_{\!r}\theta|^2 > 0$, it follows that $\kappa \geq 0$.

8.) a.) In order to find equations of state in terms of entropy (instead of the temperature) we must find the temperature as a function of the entropy. We simply solve the equation

$$\eta = c_v \log\Theta - R\log\rho + \eta_*,$$

for the temperature.

$$\eta - \eta_* = \log(\Theta^{c_v}/\rho^R) \quad \Rightarrow \quad \Theta^{c_v} = \rho^R \mathrm{e}^{(\eta-\eta_*)} \quad \Rightarrow \Theta = \rho^{R/c_v}\mathrm{e}^{(\eta-\eta_*)/c_v}.$$

This conventionally written as

$$\Theta = \rho^{(\gamma-1)}\mathrm{e}^{(\eta-\eta_*)/c_v},$$

where $\gamma = 1 + R/c_v$ is our new constant, which is actually the ratio of specific heat at constant pressure to that at constant volume. The other two equations of state are now straightforward

$$P = \rho R\rho^{\gamma-1}\mathrm{e}^{(\eta-\eta_*)/c_v} = R\rho^{\gamma}\mathrm{e}^{(\eta-\eta_*)/c_v},$$

$$\Phi = c_v\rho^{(\gamma-1)}\mathrm{e}^{(\eta-\eta_*)/c_v}.$$

b.) The governing equations of gas dynamics from the lecture notes are

$$\frac{D\rho}{Dt} + \rho\boldsymbol{\nabla}_{\!R}\boldsymbol{\cdot}\boldsymbol{V} = 0, \tag{11}$$

$$\rho\frac{D\boldsymbol{V}}{Dt} + \boldsymbol{\nabla}_{\!R}\, P - \rho\boldsymbol{F} = \boldsymbol{0}, \tag{12}$$

$$\rho\frac{D\Phi}{Dt} + P\,\boldsymbol{\nabla}_{\!R}\boldsymbol{\cdot}\boldsymbol{V} - \rho B = 0. \tag{13}$$

Simple rearrangement of conservation of mass and momentum gives the desired equations

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} (\rho \boldsymbol{V}) = 0,$$

and

$$\rho \frac{D\boldsymbol{V}}{Dt} + \boldsymbol{\nabla}_{\boldsymbol{R}} P(\rho, \eta) = \rho \boldsymbol{F}.$$

For the energy equation we use the chain rule to write

$$\rho \frac{D\Phi}{Dt} = \rho \frac{\partial \Phi}{\partial \eta} \frac{D\eta}{Dt} + \rho \frac{\partial \Phi}{\partial \rho} \frac{D\rho}{Dt},$$

and using conservation of mass in the second term gives

$$\rho \frac{D\Phi}{Dt} = \rho \frac{\partial \Phi}{\partial \eta} \frac{D\eta}{Dt} - \rho^2 \frac{\partial \Phi}{\partial \rho} \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V}.$$

Thus the energy equation becomes

$$\rho \frac{\partial \Phi}{\partial \eta} \frac{D\eta}{Dt} + \left[P - \rho^2 \frac{\partial \Phi}{\partial \rho} \right] \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V} = \rho B. \tag{14}$$

In order to reach the desired form we need to show that the term in square brackets is zero. In fact, it is a thermodynamic constraint. For an ideal gas the inequality is

$$\rho \Theta \dot{\eta} - \rho \dot{\Phi} + \mathsf{T} : \mathsf{D} \geq 0,$$

because it cannot support a heat flux. We are treating η and ρ as our independent variables, so

$$\dot{\Phi} = \frac{\partial \Phi}{\partial \eta} \dot{\eta} + \frac{\partial \Phi}{\partial \rho} \dot{\rho} = \frac{\partial \Phi}{\partial \eta} \dot{\eta} - \frac{\partial \Phi}{\partial \rho} \rho \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V},$$

by conservation of mass. Hence, the inequality becomes

$$\left[\rho \Theta - \rho \frac{\partial \Phi}{\partial \eta} \right] \dot{\eta} + \left[\rho^2 \frac{\partial \Phi}{\partial \rho} - P \right] \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V} \geq 0,$$

after using the relationship $\mathsf{T} : \mathsf{D} = -P\mathsf{I} : \mathsf{D} = -P\, \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V}$. The inequality must be satisfied for all motions and so, by the usual arguments, we have the two constraints

$$\Theta = \frac{\partial \Phi}{\partial \eta} \quad \text{and} \quad P = \rho^2 \frac{\partial \Phi}{\partial \rho}.$$

The equations of state do indeed satisfy these conditions because

$$\frac{\partial \Phi}{\partial \eta} = \frac{\partial}{\partial \eta} \left(c_v \rho^{(\gamma-1)} \mathrm{e}^{(\eta - \eta_*)/c_v} \right) = \rho^{(\gamma-1)} \mathrm{e}^{(\eta-\eta_*)/c_v} = \Theta,$$

and

$$\rho^2 \frac{\partial \Phi}{\partial \rho} = \rho^2 c_v (\gamma - 1) \rho^{(\gamma-2)} \mathrm{e}^{(\eta-\eta_*)/c_v} = R \rho^{\gamma} \mathrm{e}^{(\eta-\eta_*)/c_v} = P.$$

We could have used the equation of state directly in equation (14), but it's more fun this way and more general because it shows that the final equations do not depend on the specific equations of state for an ideal gas.
Either way, equation (14) becomes

$$\frac{\partial \Phi(\rho,\eta)}{\partial \eta}\frac{D\eta}{Dt} = B,$$

after dividing through by ρ.

c.) When there is no body heating, the energy equation becomes

$$\frac{\partial \Phi(\rho,\eta)}{\partial \eta}\frac{D\eta}{Dt} = 0.$$

We know that $\frac{\partial \Phi}{\partial \eta} = \Theta > 0$, so we must have

$$\frac{D\eta}{Dt} = 0,$$

and hence the entropy is unchanged during the motion $\eta(\boldsymbol{R},t) = \eta(\boldsymbol{R},0) = \tilde{\eta}$, a constant for all time. Thus, the remaining governing equations are

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}(\rho \boldsymbol{V}) = 0,$$

$$\rho\frac{D\boldsymbol{V}}{Dt} + \boldsymbol{\nabla}_{\boldsymbol{R}} P(\rho) = \rho \boldsymbol{F}.$$

where $P(\rho) = \rho^{\gamma} R \mathrm{e}^{(\tilde{\eta}-\eta_*)/c_v} = K\rho^{\gamma}$, where K is another constant. This argument shows that the Euler equations are appropriate for the isentropic flow of a gas, or flows for which the initial conditions are that the entropy is spatially uniform.

9.) a.) From question 8, the equations of gas dynamics in entropy form are

$$\frac{D\rho}{Dt} + \rho \boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V} = 0, \tag{15}$$

$$\rho\frac{D\boldsymbol{V}}{Dt} + \boldsymbol{\nabla}_{\boldsymbol{R}} P - \rho\boldsymbol{F} = \boldsymbol{0}, \tag{16}$$

$$\Theta\frac{D\eta}{Dt} - B = 0. \tag{17}$$

If there are no body forces or body heating $\boldsymbol{F} = \boldsymbol{0}$ and $B = 0$. The gas is at rest so $\boldsymbol{V} = \boldsymbol{0}$ and $\Theta > 0$ by definition. Hence, the equations become

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} = 0, \quad \boldsymbol{\nabla}_{\boldsymbol{R}} P = \boldsymbol{0}, \quad \frac{\partial \eta}{\partial t} = 0.$$

From the first and thirds equation $\rho(\boldsymbol{R})$ and $\eta(\boldsymbol{R})$ are both independent of time. Also, from the constitutive laws $P(\rho,\eta)$, so $P(\boldsymbol{R})$ is also independent of time. The final equation

$$\boldsymbol{\nabla}_{\boldsymbol{R}} P = \boldsymbol{0},$$

can only be satisfied if $P(t)$ is independent of spatial coordinate, so $P = P^*$ must be a constant. Hence because $P(\rho, \eta)$, it follows that

$$\rho(\boldsymbol{R}, t) = \rho^*, \quad \eta(\boldsymbol{R}, t) = \eta^*,$$

where ρ^* and η^* are constants.

b.) We pose the power series expansions

$$\rho(\boldsymbol{R}, t) = \rho^* + \epsilon\rho^{(1)} + \cdots, \quad \eta(\boldsymbol{R}, t) = \eta^* + \epsilon\Theta^{(1)} + \cdots, \quad \boldsymbol{V} = \boldsymbol{0} + \epsilon\boldsymbol{V}^{(1)} + \cdots,$$

$$P(\boldsymbol{R}, t) = P^* + \epsilon P^{(1)} + \cdots = P(\rho^*, \eta^*) + \epsilon P^{(1)} + \cdots.$$

We substitute these expansions into the governing equations to obtain

$$\frac{\partial(\rho^* + \epsilon\rho^{(1)})}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \cdot [(\rho^* + \epsilon\rho^{(1)})\epsilon\boldsymbol{V}^{(1)}] + \mathcal{O}(\epsilon^2).$$

$$(\rho^* + \epsilon\rho^{(1)})\frac{D\epsilon\boldsymbol{V}^{(1)}}{Dt} + \boldsymbol{\nabla}_{\boldsymbol{R}}[P^* + \epsilon P^{(1)}] + \mathcal{O}(\epsilon^2) = \boldsymbol{0},$$

$$\frac{D(\eta^* + \epsilon\eta^{(1)})}{Dt} = 0.$$

Recalling that ρ^* and η^* are constant and therefore $P(\rho^*, \eta^*)$ is also constant the equations become

$$\epsilon\left[\frac{\partial\rho^{(1)}}{\partial t} + \rho^*\boldsymbol{\nabla}_{\boldsymbol{R}} \cdot \boldsymbol{V}^{(1)}\right] + \mathcal{O}(\epsilon^2) = 0,$$

$$\epsilon\left[\rho^*\frac{\partial\boldsymbol{V}^{(1)}}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} P^{(1)}\right] + \mathcal{O}(\epsilon^2) = \boldsymbol{0},$$

$$\epsilon\left[\frac{\partial\eta^{(1)}}{\partial t}\right] + \mathcal{O}(\epsilon^2) = 0,$$

Hence, at $\mathcal{O}(\epsilon)$ the governing equations of linear acoustics are

$$\frac{\partial\rho^{(1)}}{\partial t} + \rho^*\boldsymbol{\nabla}_{\boldsymbol{R}} \cdot \boldsymbol{V}^{(1)} = 0, \tag{18}$$

$$\rho^*\frac{\partial\boldsymbol{V}^{(1)}}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} P^{(1)} = \boldsymbol{0}, \tag{19}$$

$$\frac{\partial\eta^{(1)}}{\partial t} = 0. \tag{20}$$

c.) We take the partial derivative with respect to time of equation (18) and the divergence of equation (19) so that

$$\frac{\partial^2\rho^{(1)}}{\partial^2 t} + \rho^*\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}_{\boldsymbol{R}} \cdot \boldsymbol{V}^{(1)}\right) = 0,$$

and

$$\rho^*\boldsymbol{\nabla}_{\boldsymbol{R}} \cdot \frac{\partial\boldsymbol{V}^{(1)}}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}}^2 P^{(1)} = 0.$$

Subtracting these two equations and using the fact that temporal and spatial derivatives commute gives

$$\frac{\partial^2 \rho^{(1)}}{\partial^2 t} - \boldsymbol{\nabla}_{\boldsymbol{R}}^2 \, P^{(1)} = 0. \tag{21}$$

We can express the density perturbation in terms of the pressure perturbation by using the form of the constitutive relation, $P(\rho, \eta)$,

$$P^* + \epsilon P^{(1)} = P(\rho^* + \epsilon\rho^{(1)}, \eta^* + \epsilon\eta^{(1)}) = P(\rho^*, \eta^*) + \epsilon\rho^{(1)} P_\rho|_{(\rho^*,\eta^*)} + \epsilon\eta^{(1)} P_\eta|_{(\rho^*,\eta^*)},$$

after Taylor expanding the arguments of the pressure. Thus,

$$P^{(1)} = \rho^{(1)} P_\rho^* + \eta^{(1)} P_\eta^*,$$

and taking the second derivative with respect to time give

$$\frac{\partial^2 P^{(1)}}{\partial t^2} = P_\rho^* \frac{\partial^2 \rho^{(1)}}{\partial t^2} + P_\eta^* \frac{\partial \eta^{(1)}}{\partial t^2}.$$

The energy equation (20) states that $\frac{\partial \eta^{(1)}}{\partial t} = 0$, so

$$\frac{\partial^2 P^{(1)}}{\partial t^2} = P_\rho^* \frac{\partial^2 \rho^{(1)}}{\partial t^2}.$$

Hence the equation (21) becomes

$$\frac{\partial^2 P^{(1)}}{\partial^2 t} - P_\rho^* \, \boldsymbol{\nabla}_{\boldsymbol{R}}^2 \, P^{(1)} = 0, \quad \Rightarrow \quad \frac{\partial^2 P^{(1)}}{\partial^2 t} = P_\rho^* \, \boldsymbol{\nabla}_{\boldsymbol{R}}^2 \, P^{(1)},$$

which is the desired wave equation. The wave speed is given by

$$c^2 = \left. \frac{\partial P}{\partial \rho} \right|_{(\rho^*,\eta^*)},$$

and this is c called the speed of sound.