MATH45061: EXAMPLE SHEET[1] VI

1.) Show that the Eulerian rate of deformation tensor D can be decomposed into the form

$$\mathsf{D} = \widetilde{\mathsf{D}} + \frac{1}{3}(\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V})\,\mathsf{I},$$

where $\widetilde{\mathsf{D}}$, the deviatoric part of the tensor, is such that $\mathrm{trace}(\widetilde{\mathsf{D}}) = 0$.

2.) Show that for a compressible Newtonian fluid in which the heat flux is given by $\boldsymbol{Q} = -\kappa\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta$, the Clausis–Duhem inequality is satisfied by every admissible thermomechanical process if and only if

$$\mu \geq 0, \quad \lambda + \frac{2}{3}\mu \geq 0, \quad \kappa \geq 0.$$

Hint: You will find it useful (necessary) to separate the rate of deformation tensor, D, into deviatoric and non-deviatoric parts to ensure the terms in your final expression are independent.

3.) An incompressible Newtonian fluid moves within a fixed domain $\mathcal{D}$ with stationary walls and is subject to a conservative body force $\boldsymbol{F} = -\boldsymbol{\nabla}_{\boldsymbol{R}}G$. Show that

$$\frac{DK}{Dt} + 2\mu\int_{\mathcal{D}} \mathsf{D} : \mathsf{D}\,\mathrm{d}\mathcal{V} = 0,$$

where $K(t)$ is the kinetic energy of the fluid.
Hint: You will need to use the balance of linear momentum equation and the divergence theorem after converting the integral into the appropriate form.

4.) An incompressible fluid is placed between two concentric circular cylinders of radii a and b, where $b > a$. The inner cylinder is stationary, but the outer cylinder is rotated at a constant angular velocity Ω. A plausible velocity field is of the form $\boldsymbol{V} = F(R)\boldsymbol{e}_{\Theta}$, where (R, Θ, Z) is cylindrical polar coordinate system chosen such that the two cylinders are located at $R = a$ and $R = b$; and $\boldsymbol{e}_{\Theta}$ is a unit vector in the Θ direction. We shall work entirely within the Eulerian framework for this question, so we can dispense with the overbars for ease of notation.

a.) Show that if we define Eulerian coordinates $\xi^1 = R$, $\xi^2 = \Theta$, $\xi^3 = Z$, the contravariant components of the velocity field are

$$V^1 = 0, \quad V^2 = \frac{F(R)}{R}, \quad V^3 = 0.$$

b.) The fluid obeys a general non-Newtonian constitutive law

$$\mathsf{T} = -P\mathsf{I} + 2\,\mu({}_{\mathsf{D}}I_2)\,\mathsf{D},$$

so that the viscosity is not necessarily constant, show that

$$T^{ij} = \begin{pmatrix} -P & \mu\,(F(R)/R)' & 0 \\ \mu\,(F(R)/R)' & -P/R^2 & 0 \\ 0 & 0 & -P \end{pmatrix},$$

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where the prime $'$ denotes the derivative with respect to R and for ease of notation the dependence of μ on ${}_{\mathsf{D}}I_2$ has been suppressed.

c.) The fluid is influenced by a gravitational body force aligned with the axis of the cylinder $\boldsymbol{F} = -g\,\boldsymbol{e}_z$. By using the balance of linear momentum equation, show that an axisymmetric (independent of Θ) solution is given by

$$T^{11} = R^2 T^{22} = T^{33} = -P = P_0 - \rho g Z - \int^R \rho \frac{(F(R))^2}{s}\,\mathrm{d}s,$$
$$\text{and} \quad T^{12} = K/R^3,$$

where P_0 and K are constants. What is the corresponding physical component of the stress $\sigma_{(12)}$?

d.) Explain why the traction exerted on the fluid at radius R is given by

$$\boldsymbol{T} = -P\boldsymbol{e}_R + \frac{K}{R^2}\boldsymbol{e}_\Theta.$$

Hence, show that $K = T/2\pi$, where $T\,\boldsymbol{e}_Z$ is the total torque per unit length exerted on the cylinder.

e.) If the fluid is a power law fluid then $\mu({}_{\mathsf{D}}I_2) \propto (|{}_{\mathsf{D}}I_2|)^{(n-1)/2}$. Show that

$$T^{12} = \alpha R^{n-1}\left[\frac{\partial(F/R)}{\partial R}\right]^n,$$

for some constant α, and hence that

$$\Omega = \frac{n}{2}\left(\frac{T}{2\pi\alpha}\right)^{\frac{1}{n}}\left(a^{-\frac{2}{n}} - b^{-\frac{2}{n}}\right).$$

5.) For an incompressible Newtonian fluid, the internal energy can only be a function of temperature; if we assume small temperature variations then $\Phi = c_p\Theta$, where the constant c_p is called the heat capacity of the fluid.

a.) Show that the energy equation in Cartesian coordinates

$$\rho c_p\left[\frac{\partial\Theta}{\partial t} + V_I\Theta_{,I}\right] = \mu\left(V_{I,J}V_{I,J} + V_{J,I}V_{I,J}\right) + \kappa\Theta_{,KK} + \rho B.$$

b.) In the absence of body heating, show that a dimensionless form of the equation is given by

$$Pe\left[\frac{\partial\Theta^*}{\partial t^*} + V_I^*\Theta^*_{,I}\right] = \Theta^*_{,KK} + PeC\left(V^*_{I,J}V^*_{I,J} + V^*_{I,J}V^*_{J,I}\right),$$

where the starred quantities are dimensionless and Pe and C are dimensionless groups to be found. **Hint:** Choose the natural timescale.

c.) In general $C \ll 1$ and we shall neglect it by setting $C = 0$. If we have a steady two-dimensional, shear flow above a heated, plane wall located at $Y^* = 0$, such that the two Cartesian velocity components are $U^* = \gamma Y^*$, $V^* = 0$, show that when $Pe \gg 1$, the thermal boundary layer has thickness $\mathcal{O}(Pe^{-1/3})$, and that the dimensionless energy equation reduces to a boundary layer equation

$$\gamma\,\tilde{Y}\frac{\partial\tilde{\Theta}}{\partial\tilde{X}} = \frac{\partial^2\tilde{\Theta}}{\partial\tilde{Y}^2}.$$